

**MATHEMATICS 3:  
CALCULUS**

**PR. RALPH W.P.  
MASENGE**

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# Mathematics 3

Calculus

Prepared by Pr. Ralph W.P. Masenge

**African Virtual university**

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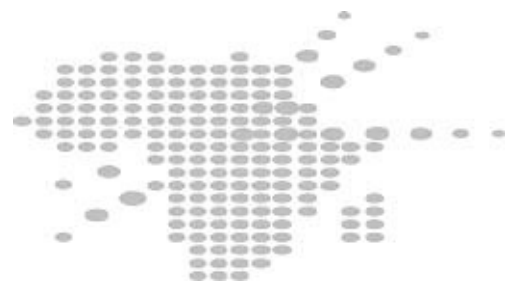


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## Table of Contents

I.

Mathematics 3, Calculus	3
II. Prerequisite Course or Knowledge	3
III. Time	4
IV. Materials	4
V. Module Rationale	5
VI. Content	6
6.1 Overview	6
6.2 Outline	6
6.3 Graphic Organizer	8
VII. General Objective(s)	9
VIII. Specific Learning Objectives	9
IX. Teaching And Learning Activities	10
9.1 Pre-Assessment	10
9.2 Pre-Assessment Answers	17
9.3 Pedagogical Comment For Learners	18
X. Key Concepts (Glossary)	19
XI. Compulsory Readings	26
XII. Compulsory Resources	27
XIII. Useful Links	28
XIV. Learning Activities	31
XV. Synthesis Of The Module	77
XVI. Summative Evaluation	120
XVII. Main Author of the Module	131



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**I.**

## **Mathematics 3, Calculus**

Prof. Ralph W.P.Masenge, Open University of Tanzania

**Figure 1** : Flamingo family curved out of horns of a Sebu Cow-horns

**II.**

### **Prerequisite Courses or Knowledge**

#### **Unit 1: Elementary differential calculus (35 hours)**

Secondary school mathematics is prerequisite. Basic Mathematics 1 is co-requisite.

This is a level 1 course.

#### **Unit 2: Elementary integral calculus (35 hours)**

Calculus 1 is prerequisite.

This is a level 1 course.

#### **Unit 3: Sequences and Series (20 hours)**

Priority A. Calculus 2 is prerequisite.

This is a level 2 course.

## Unit 4: Calculus of Functions of Several Variables (30 hours)

Priority B. Calculus 3 is prerequisite.

This is a level 2 course.



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### III.

#### Time

120 hours

### IV.

#### Material

The course materials for this module consist of:

Study materials (print, CD, on-line)

(pre-assessment materials contained within the study materials)

Two formative assessment activities per unit (always available but with specified submission date). (CD, on-line)

References and Readings from open-source sources (CD, on-line)

ICT Activity files



Those which rely on copyright software

Those which rely on open source software

Those which stand alone

Video files

Audio files (with tape version)

Open source software installation files

Graphical calculators and licenced software where available

( *Note*: exact details to be specified when activities completed)

**Figure 2** : A typical internet café in Dar Es Salaam



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**V.**

### **Module Rationale**

The secondary school mathematics syllabus covers a number of topics, including differentiation and integration of functions. The module starts by introducing the concept of limits, often missed at the secondary school level, but crucial in learning these topics. It then uses limits to define continuity, differentiation and integration of a function. Also, the limit concept is used in discussing a class of special functions called sequences and the related topic of infinite series.



## **VI.**

### **Content**

#### **6.1**

##### **Overview**

This is a four unit module. The first two units cover the basic concepts of the differential and integral calculus of functions of a single variable. The third unit is devoted to sequences of real numbers and infinite series of both real numbers and of some special functions. The fourth unit is on the differential and integral calculus of functions of several variables.

Starting with the definitions of the basic concepts of limit and continuity of functions of a single variable the module proceeds to introduce the notions of differentiation and integration, covering both methods and applications.

Definitions of convergence for sequences and infinite series are given. Tests for convergence of infinite series are presented, including the concepts of interval and radius of convergence of a power series.

Partial derivatives of functions of several variables are introduced and used in formulating Taylor's theorem and finding relative extreme values.

#### **6.2**

##### **Outline**

##### **Unit 1: Elementary differential calculus (35 hours)**

Level 1. Priority A. No prerequisite. Basic Mathematics 1 is co-requisite.

Limits (3)

Continuity of functions. (3)

Differentiation of functions of a single variable. (6)

Parametric and implicit differentiation. (4)

Applications of differentiation. (6)

Taylor's theorem. (3)

Mean value theorems of differential calculus. (4)

Applications. (6)

## **Unit 2: Elementary integral calculus (35 hours)**

Level 1. Priority A. Calculus 1 is prerequisite.

Anti derivatives and applications to areas. (6)

Methods of integration. (8)

Mean value theorems of integral calculus. (5)



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Numerical integration. (7)

Improper integrals and their convergence. (3)

Applications of integration. (6)

## **Unit 3: Sequences and Series (20 hours)**

Level 2. Priority A. Calculus 2 is prerequisite.

Sequences (5)

Series (5)

Power series (3)

Convergence tests (5)

Applications (2)

## **Unit 4: Calculus of Functions of Several Variables (30 hours)**

Level 2. Priority B. Calculus 3 is prerequisite.

Functions of several variables and their applications (4)

Partial differentiation (4)

Center of masses and moments of inertia (4)

Differential and integral calculus of functions of several variables:

Taylor's theorem (3)

Minimum and Maximum points (2)

Lagrange's Multipliers (2)

Multiple integrals (8)

Vector fields (2)



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## 6.3

### Graphic Organizer

This diagram shows how the different sections of this module relate to each other.

The central or core concept is in the centre of the diagram. (Shown in red).

Concepts that depend on each other are shown by a line.

*For example:* Limit is the central concept. Continuity depends on the idea of Limit. The Differentiability depend on Continuity.

Functions

Functions

of a single

of several

variable

variable

Differentiability

Limit

Continuity

Integrability

Sequence

Infinite

Series



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## **VII. General objective(s)**

You will be equipped with knowledge and understanding of the properties of elementary functions and their various applications necessary to confidently teach these subjects at the secondary school level.

You will have a secure knowledge of the content of school mathematics to confidently teach these subjects at the secondary school level

You will acquire knowledge of and the ability to apply available ICT to improve the teaching and learning of school mathematics

## **VIII. specific learning objectives**

### **(Instructional objectives)**

You should be able to demonstrate an understanding of

- The concepts of limits and the necessary skills to find limits.
- The concept of continuity of elementary functions.
- ... and skills in differentiation of elementary functions of both single and several variables, and the various applications of differentiation.
- ... and skills in integration of elementary functions and the various applications of integration.
- Sequences and series, including convergence properties.

You should **secure your knowledge** of the following school mathematics:

- Graphs of real value functions.
- Idea of limits, continuity, gradients and areas under curves using graphs of functions.
- Differentiation and integration a wide of range of functions.
- Sequences and series (including A.P., G.P. and  $\sum$  notation).
- Appropriate notation, symbols and language.



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**IX.**

**Teaching and learning activities**

**9.1**

**Pre-assessment**

**Module 3: Calculus**

**Unit 1: Elementary Differential Calculus**

1. Which of the following sets of ordered pairs  $(x, y)$  represents a function?

(a)

(1,1), (1,2), (1,2), (1,4)

(b)

(1,1), (2,1), (3,1), (4,1)

(c)

(1,1), (2,2), (3,1), (3,2)

(d)

(1,1), (1,2), (2,1), (2,2)

{(a)

| | ( ) b

}

The set which represents a function is

| (c)

| (d)

2. The sum of the first n terms of a GP, whose first term is a and common

$n+1$

$[1 - r$

$]$

ratio is r, is  $S = a$

. For what values of r will the GP converge?

n

| 1 r |

-

[

]

$$\{(a) r < -1$$

$$\mid ()$$

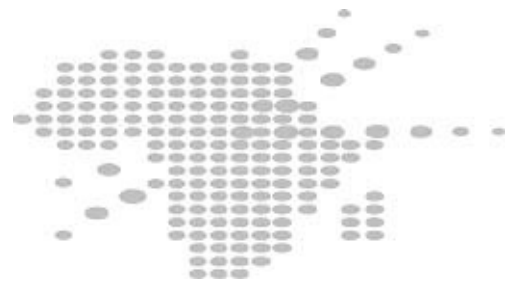
b

$$r = 1$$

The GP will converge if

$$\mid \} \mid (c) r > 1$$

$$\mid \} \mid (d) r < 1$$



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3. Find the equation of the tangent to the curve

2

$$y = 2x - 3x + 2 \text{ at } (2, 4) .$$

$$\{(a) y = 5x + 6$$

$$\mid \mid (b) y = 6x - 5$$

The equation of the tangent at (2,4) is  $\} \mid (c) y = 5x - 6$

$$\mid \} \mid (d) y = 6x + 5$$

$\frac{d^2 y}{dx^2}$

$\frac{dy}{dx}$

4. Given the function  $y = \sin(x) + \cos(x)$ ,

find

+ 2

- y .

$\frac{dx^2}{dx^2}$



dx

(a)  $-4\sin(x)$

(b)  $4\cos(x)$

The value of the expression is (c)  $4\sin(x)$

(d)  $-4\cos(x)$

5. Using Maclaurin's series expansion, give a cubic approximation of  $y = \tan(x)$ .

[

1 3

|

$1 + x + x$

3

(a)

1

|

3

x

x

|

-

( )

b

The required cubic is

3

(c)

1

|

$$1 - x +$$

3

x

|

3

(d)

|

1 3

|

$$x + x$$

|

3



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**Unit 2:**

**Elementary Integral Calculus**

∫ (a) Non - existent

∫ 2

x

|

- 2

$$x \geq 2$$

| ( )

b

- 2

6. If  $f(x) = \begin{cases}$

then,  $\lim_{x \rightarrow 2} f(x) = \end{cases}$

$\begin{cases} x - 4$

$x < 2$

$x \rightarrow 2$

| (c)

2

| (d)

0

|

x

(a)

x

|

df

|

x-

b

( )

x 1

7. If

x

$$f(x) = x$$

Then

$$= \int$$

x

$$dx \int (c) x^{1+\ln(x)}$$

|

x

$$\int (d)$$

$$\ln(x)x$$

8. The anti-derivative of a function  $f(x)$  is defined as any function whose derivative is  $f(x)$ . Therefore, the anti-derivative of

$-x$

$$f(x) = \sin(x) + e$$

$$\int (a) - \cos(x) + -x$$

e

$$\int (b) \cos(x) - -x$$

e

$$\text{is } F(x) = \int$$

$-x$

$$\int (c)$$

$$\cos(x) + e$$

|

$-x$

$$\int (d) - [\cos(x) + e]$$

9. A trapezium (also known as a trapezoid) is any quadrilateral with a pair of

opposite sides being parallel. If the lengths of the sides of a trapezium are  $f$

0

and  $f$ , and if the distance between the pair of parallel sides is  $h$ , then the

1

[

(

$h f$

$f$ )

0 +

1

| (a)  $h$

( )

( $f$

$f$

0 -

1)

| |  $b$

area of the trapezium is

Area =

2

} (c) (

$h f$

$f$ )

|

0 -

1

$| (d) h (f$

f

0 +

1 )

| l

2



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10. If the points  $A(- ,$   
 $h f ) , B (0, f ) , C ( ,$   
 $h f ) ,$  lie on a parabola

-1

0

1

$y = ax^2 + bx + c ,$  then, it can be shown that:

$f - 2 f + f$

$f - f$

1

0

1

$A = -$

, B

1

0

=

, and  $C = f$ .

2

$2h$

$2h$

0

Then, the area under the parabola that lies between the ordinates at  $x = -h$

$\int$

$h (f_4$

$1 +$

$f_0 + f$

$-$

$1)$

$|$

$2$

$| (a) h (f_4$

$1 -$

$f_0 + f$

$h$

$b$

$( )$

$-$

$1)$

||

and  $x = h$  is given by

$\int y dx =$

3

}

-h

(c) h

|

(f

4

1 +

$f_0 + f$

-

1)

| (d 3

)

|

h (f 4

1 +

$f_0 + f$

-

1)

| l

2

### Unit 3: Sequences and Series



2  
3  
4  
5

11. The first four terms of a sequence  $\{a_n\}$  are ,

,  
,  
.  
 $n\}$

3  
5  
7  
9  
[

28  
|(a)

47  
28  
| |()  
b

Therefore, a

57  
27 =  
{(c) 28  
|

49

| (d) 28

| l

55



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[ (a) -1

|

1

| ( )

b

1

12. The limit L of a sequence with a

is

L =

n =

(

n n

}

+1

(c) 0

| |

1

(d)

2

13. The sequence whose  $n$ -th term is given by a

is:

$n = ($

$)n 1$

$1 +$

$-$

(a) convergent

(b)

(c) increasing

(d) divergent

(e) decreasing

14. If  $\{a_n\}$  is a sequence of real numbers, and if  $\lim_{n \rightarrow \infty} a_n = L$ , then the infinite

series

$\sum_{n=1}^{\infty} a_n$

converges if

(a)  $L < \infty$

(b)

(c)

(d)

(e)

$L < 1$

series  $\sum a_n$  converges only if

k

{ (c) L 0

k

=

=1

| | (d) L <1

[

1

|

1- n

| (a)

1

n

1

1

|

1

|

-

b

()

15. If

S

then

n +

S

1

$$n = \sum ($$

)

| -

|

$$n = \{ () 1$$

1

k

c

1

k n 1

= \

+ )

|

-

|

n n

(d)

+1

|

1

1

|

-

l

n + 1 n



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**Unit 4: Calculus of Functions of Several Variables**

16. The area enclosed by the curve  $y = 4$  and

2

$y = 1 + x$  is

(a) = 3 20

(b) = 7

(c) = 0

(d) = 20 3

17. If  $f ( ,$

$x y) = \ln( 3$

2

2

3

$x - x y + y )$  and  $g(t) = e^{-t} \sin(t) ,$

t h e n

$g(f (x, y))$  is given by:

$\sin( 3$

2

2

3

$x - x y + y )$

(a) =  $\ln( 3 2 2 3$

$x - x y + y )$

$\sin[ \ln( 3$

2

2

3

$x - x y + y ) ]$

(b) =

$\ln( 3$

2

2

3

$x - x y + y )$

$\sin[ \ln( 3$

2

2

3

$x - x y + y ) ]$

(c) =

$( 3 2 2 3$

$x - x y + y )$

$( 3 2 2 3$

$x - x y + y )$

$$(d) = \sin[\ln(3 \cdot 2 \cdot 2 \cdot 3)]$$

$$x - x y + y ]$$

18. The volume **V** of an ideal gas depends on (**is a function of**) two independent variables, namely, temperature **T** and pressure **P**. Specifically, **V** is directly proportional to **T** but inversely proportional to **P**. Assume that for some unspecified temperature and pressure, the volume **V** is **100** units. If one then doubled the pressure and halves the temperature, then **V** becomes:

- (a) 200 units
- (b) 100 units
- (c) 50 units
- (d) 25 units



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19. The domain **D** and range **R** of the function

2

2

f ( ,

$$x y) = \sin 9 - x - y$$

are:

$$\left[ (a) \Rightarrow D = ($$

$$\{ ,x y)9 < 2x + 2y\}; R = (- )1,$$

1

$$\left| \left| ( )b \Rightarrow D = (\{ ,x y)9 \leq 2x + 2y\}; R = [- ]1,$$



1

{ (c) ⇒ D = (

2

|

{ ,x y)9 ≥ x + y };R =[- ]1,

1

| (d) ⇒ D = ({ ,x y)9 > 2x + 2

y };R (

)

1

,

1

{

= -

20. If  $f(x, y) = 2xy^3 + x^2y^2 - 3yx^3 + 4y$  and one denotes the limit

$[ f(x + h, y) - f(x, y)$

$\partial f$

lim

] with

, then

$h \rightarrow 0$

$h$

$\partial x$

{ |

dy

dy

dy

dy

6 2

xy

2 2

x y

3 3

x

4

| (a)

+

-

+

dx

dx

dx

dx

∂f

| | ( )

b

dy

dy

dy

dy

=

3

2

2

2

2

3

{

$$2y + 6xy$$

$$+ 2xy + 2x y$$

$$- 9x y - 3x$$

$$+ 4$$

$$\partial x$$

c

$$dx$$

$$dx$$

$$dx$$

$$dx$$

|()

3

2

2

|

$$2y + 2xy - 9x y$$

(d)

|

6 2

$xy - 3 3$

$x + 4$

| |



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## 9.2

### Pre-Assessment Answers

1. (b)

2. (d)

3. (c)

4. (a)

5. (b)

6. (a)

7. (c)

8. (b)

9. (d)

10. (a)

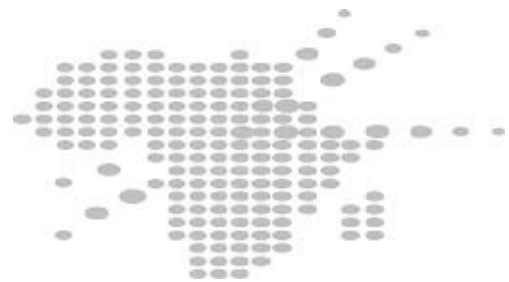
11. (d)

12. (b)

13. (c)

14. (c)

- 15. (b)
- 16. (d)
- 17. (c)
- 18. (d)
- 19. (b)
- 20. (c)



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### 9.3

#### **Pedagogical Comment For Learners**

Preassessment is an important self-assessment exercise. You are strongly encouraged to solve all the questions. Each correct solution is worth 5 Marks, giving a total of 100 Marks.

Preassessment has two main objectives, which are: to indicate to the learner knowledge need before embarking on the module and to link the known material with material to be learnt in the course of the module.

The learner is advised to solve the problems sequentially unit by unit, starting with Unit 1 and ending with Unit 4 as depicted in the following flowchart.

**Unit 1 ⇒ Unit 2 ⇒ Unit 3**  
**⇒ Unit 4**

The serial coverage is recommended because material covered in one unit informs the contents of the unit that comes after it.

The learner should resist the temptation of working backwards from the solutions

given in the solution key. Verifying a solution may hide one's lack of knowledge of some basic concepts, leading to poor understanding of the contents of subsequent learning activities.

Marks scored in the preassessment give an indication of the learner's degree of preparedness to embark on the learning activities. A below average score (0 – 40%) may indicate the need for revising some prerequisite knowledge before proceeding to the learning activities. An average score (41 – 60%) signifies the readiness of the learner to embark on the module with occasional cross reference to some prerequisite materials. An above average score (61 – 100%) is a clear indication that the learner is ready to confidently embark on the module.



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X.

## Key Concepts (Glossary)

### Limit of a function

A function  $f(x)$  has a limit  $L$  as  $x$  approaches point  $c$  if the value of  $f(x)$  approaches  $L$  as  $x$  approaches  $c$ , on both sides of  $c$ . We write

$\lim$

$f(x) = L$

$x \rightarrow c$

### One sided limits

A function may have different limits depending from which side one approaches  $c$ . The limit obtained by approaching  $c$  from the right (values greater than  $c$ ) is

called the right-handed limit. The limit obtained by approaching  $c$  from the left

(values less than  $c$ ) is called the left-handed limit. We denote such limits by

$\lim$

$\lim$

+

$f(x) = L,$

–

$f(x) = L$

+

$x \rightarrow c$

–

$x \rightarrow c$

### **Continuity of a function**

A function  $f(x)$  is said to be continuous at point  $c$  if the following three conditions are satisfied:

- The function is defined at the point, meaning that  $f(c)$  exists,
- The function has a limit as  $x$  approaches  $c$ , and
- The limit is equal to the value of the function. Thus for continuity we must

$\lim$

have

$f(x) = f(c).$

$x \rightarrow c$

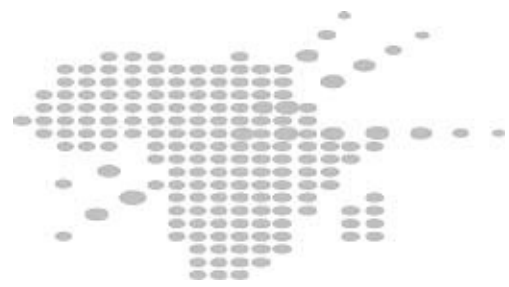
### **Continuity over an Interval**

A function  $f(x)$  is continuous over an interval  $I = [a, ]$

$b$  if it is continuous at

each interior point of  $I$  and is continuous on the right hand at  $x = a$  and on the left hand at  $x = b$ .

b



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## Discontinuity

A function has a discontinuity at  $x = c$  if it is not continuous at  $x = c$ .

$\sin x$

For example,  $f(x) =$

has a discontinuity at  $x = 0$ .

$x$

## Jump discontinuity

A function  $f$  is said to have a jump discontinuity at

$x = c$  if

$f(x) = M$

$\lim_{x \rightarrow c^-}$

,

$f(x) = L$

$\lim_{x \rightarrow c^+}$

, but  $L \neq M$ .

$x \rightarrow c^-$

$\rightarrow$

$x \rightarrow c^+$



→

**Removable discontinuity**

A function  $f(x)$  has a removable discontinuity at  $x = c$  if either  $f(c)$  does not exist or  $f(c) \neq L$

**Partial derivative**

A partial derivative of a function of several variables is the derivative of the given function with respect to one of the several independent variables, treating all the other independent variables as if they were real constants. For, example,

if  $f(x,y,z) = x^2 y + 3xz^2 - xyz$  is a function of the three independent variables,  $x$ ,  $y$ , and  $z$ , then the partial derivative of  $f(x)$  with respect to the variable  $x$  is the

function.  $g(x,y,z) = 2xy + 3z^2 - yz$

**Critical points**

A value  $x = c$  in the domain of a function  $f(x)$  is called a critical point if the derivative of  $f$  at the point is either zero or is not defined.

**Critical values**

The value of a function  $f(x)$  at a critical point of  $f$  is called a critical value.



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**Derivative of a function**

The derivative of a function  $y = f(x)$  is defined either as

$$\frac{f(x) - f(c)}{x - c}$$

$$\frac{f(c+h) - f(c)}{h}$$

$$\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

lim

]

]

|

|

or as  $\lim$

| provided that

x

→

[

h

→ c

[

x - c

]

h

0

]

the limit exists.

### **Differentiable**

A function  $f(x)$  is said to be differentiable at

$f(x) - f(c)$

$x = c$  if

[

lim

] exists.

$$x \rightarrow c$$

$$x - c$$

### **Implicitly defined functions**

A function  $y$  is said to be implicitly defined as a function of  $x$  if  $y$  is not isolated on one side of the equation. For example, the equation

2

$$xy - 2 = 2$$

3

$$xy + x = 3$$

defines  $y$  implicitly as a function of  $x$ .

### **Implicit differentiation**

Implicit differentiation is a method of differentiating a function that is defined implicitly, without having to solve the original equation for  $y$  in terms of  $x$ .

### **Necessary condition**

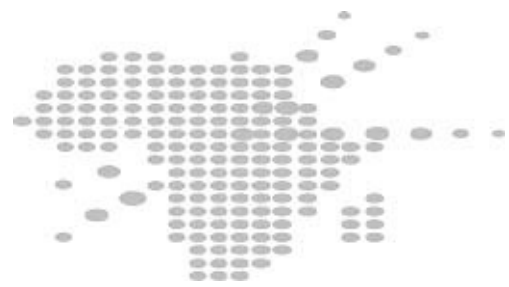
$P$  is said to be a necessary condition for  $Q$  if whenever  $P$  is not true then  $Q$  is also not true. For example, continuity ( $P$ ) is a necessary condition for differentiability ( $Q$ ). If a function is not continuous then it is not differentiable. In brief, one says:

$P$  is implied in  $Q$ , written as  $P \leftarrow Q$ .

### **Necessary and sufficient condition**

$P$  is said to be a necessary and sufficient condition for  $Q$  if  $P$  implies  $Q$  and  $Q$  implies  $P$ . For example, If a triangle is equilateral ( $P$ ), then its three angles are equal ( $Q$ ), and if the three angles of a triangle are equal ( $Q$ ) then the triangle is equilateral ( $P$ ).

One writes  $P \leftrightarrow Q$



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## Slope of Tangent

If a function  $y = f(x)$  is differentiable at  $x = c$  then the slope of the tangent to the graph of  $f$  at the point  $(c, f(c))$  is  $f'(c)$ .

## Sufficient condition

P is said to be a sufficient condition for Q if, whenever P is true then Q is also true. In other words, P implies Q. For example, differentiability (P) is a sufficient condition for continuity (Q).

## Sequence

A sequence is an unending list of objects (real numbers)  $a_1, a_2, a_3, a_4, \dots$ . The

1

2

3

4

number  $a_n$  is called the  $n$ th

term of the sequence, and the sequence is denoted

$\{a_n\}$

by  $\{a_n\}$ .

$n\}$

## Convergence of a Sequence

A sequence  $\{a_n\}$  is said to converge if there exists a limit  $L$  such that

$\lim_{n \rightarrow \infty} a_n = L$

$n\}$

$n$

$n \rightarrow \infty$

If the limit does not exist then it is said to diverge (or is divergent).

### **Infinite Series**

$\infty$

An infinite series, written, is a sum of elements of a sequence

$a_k$

$\sum$

$\{ a .$

$n\}$

$k = 1$

### **Partial Sums**

The  $t$

$n$  h partial sum of a series, written  $S$  is the sum of the first  $n$  terms:  $n$

$$S = a + a + a + \dots + a .$$

$n$

1

2

3

$n$

### **Arithmetic series**

An arithmetic series is a series in which the difference between any two consecutive terms is a constant number  $d$  . If the first term of the series is  $a$  then the

$t$

$n$ th partial sum of such a series is given by

$n$

$S =$

$a + n - d$

$n$

(2

(

1))

2



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### **Geometric series**

A geometric series is a series in which, except for the first term, each term is a constant multiple of the preceding term. If the first term is  $a$  and the common factor is  $r$  then the  $t$

$n$ th partial sum of the geometric series is given by.

$n+1$

[  $1 - r$

$S = a$

]

$n$

[  $r - 1$  ]

-

[  
]

**Power series**

A power series in  $x$  is an infinite series whose general term involves a power of the continuous variable  $x$ .

$\infty$

For example,

$a$

$a x$

$a x^2$

...

$a x^k$

$a x^k$

$0 +$

$1$

$+ 2$

$2$

$+$

$3$

$3$

$+ = \sum_{k=0}^{\infty} a_k x^k$  is a power series.

$k=0$

**Interval of convergence of a power series**

The interval of convergence of a power series is an interval (finite or infinite)

$- R < x < R$  in which the series converges.

## Radius of convergence of a power series

The radius of convergence of a power series is the number  $R$  appearing in the interval of convergence.

## Definite integral

The definite integral of a **continuous** function  $f(x)$ , defined over a **bounded**

$n$

**interval**  $[a, b]$ , is the **limit of a sequence of Riemann sums**  $S =$

$x$

$\Delta f(\xi$

$n$

$\sum_{i=1}^n f(\xi_i) \Delta x_i$ )

$i=1$

obtained by subdividing (partitioning) the interval  $[a, b]$  into a number  $n$  of

subintervals  $\Delta x_i = x_i - x_{i-1}$  in such a way that as  $n \rightarrow \infty$  the largest subinterval in

$i$

$i$

$i-1$

the sequence of partitions also tends to zero. The definite integral of  $f(x)$  over

$[a, b]$  is denoted by  $\int_a^b f(x) dx$

$\int_a$





An improper integral is a definite integral in which either the interval  $[a, b]$  is

not bounded  $a = -\infty$ ,  $b = \infty$  the function (integrand)  $f(x)$  is discontinuous at some points in the interval, or both.

## Domain and Range

If  $z = f(x, y)$  is a function of two independent variables then the set

$D = \{ (x, y) \in \mathbb{R}^2 \}$  of points for which the function  $z = f(x, y)$  is defined is

called the **Domain of z**, and the set  $R = \{z = f(x, y) : (x, y) \in D\}$  of values of the function at points in the domain is called the **Range of z**.

## Graph of a function

The graph of a function  $z = f(x, y)$  is the set  $G = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D\}$ .

While the graph of a function of a single variable is a curve in the  $xy$  - plane, the graph of a function of two independent variables is usually a surface.

## Level curve

A level curve of a function of two variables  $z = f(x, y)$  is a curve with an equation of the form  $f(x, y) = C$ , where  $C$  is some fixed value of  $z$ .

## Limit at a point

Let  $(a, b)$  be a fixed point in the domain of the function  $z = f(x, y)$  and  $L$  be a real number.  $L$  is said to be the limit of  $f(x, y)$  as  $(x, y)$  approaches  $(a, b)$  written

$\lim_{(x, y) \rightarrow (a, b)} f(x, y) = L$ , if for every small number  $\epsilon > 0$  one can find

$(x, y) \rightarrow (a, b)$

a corresponding number  $\delta = \delta(\epsilon)$

$\in \mathbb{R}$  such that  $f(x, y) - L < \epsilon$  whenever

$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$ .

)

$b < \delta$ .

One also writes

$\lim_{(x, y) \rightarrow (a, b)} f(x, y) = L$  or  $f(x, y)$  approaches  $L$  as  $(x, y)$  approaches  $(a, b)$ .

$x \rightarrow a$

$y \rightarrow b$



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## Continuity

A function  $z = f(x, y)$  is continuous at a point  $(a, b) \in D$  if three conditions are satisfied:

(i)  $f(x, y)$  is defined at  $(a, b)$

$f(a, b)$  [the value  $f(a, b)$  exists],

(ii)  $f(x, y)$  has a limit  $L$  as  $(x, y)$  approaches  $(a, b)$ ,

$L = f(a, b)$ , and

(iii) The limit  $L$  and the value  $f(a, b)$

of the function are equal.

This definition can be summed up by writing  $\lim$

$= f(a, )$

$b .$

$(x, y) \rightarrow (a, b)$

### **Center of gravity**

The center of gravity is also referred to as the center of mass. This is a point at which vertical and horizontal moments of a given system balance.

### **Taylor's formula**

This seeks to extend the Taylor series expansion of a function  $f(x)$  of a single variable at a point  $x = a$  to a function  $f(x, y)$  of two variables at a point  $(a, b)$ .

### **Relative Extrema**

Relative extrema is a collective terminology for the relative maximum and minimum values of a function. The singular form of the word is relative extremum, which may be a relative maximum or a relative minimum value.

### **Lagrange Multipliers**

Lagrange multipliers are the numbers (or parameters) associated with a method known as the Lagrange multipliers method for solving problems of optimization (extrema) subject to a given set of constraints.



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**XI.**

### **Compulsory Readings**

**The Calculus Bible, Prof. G.S. Gill:** Brigham Young University, Maths Depart-

ment. Brigham Young University – USA.

- Chapter 2: Limits and Continuity, pp 35 – 94.
- Chapter 6: Techniques of integration, pp 267-291
- Chapter 8

*Rationale/Abstract:* This is a complete open source textbook in Calculus. Following the complete book will provide a comprehensive resource to support this course. Specific references to sections of the book are given in the learning activities to provide activities, readings and exercises.



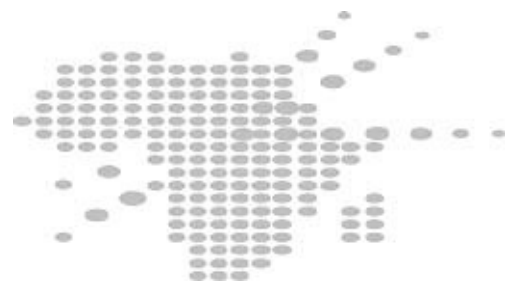
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## **XII. Compulsory Resources**

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**wxMaxima.** This is Computer Algebra System (CAS). You should double click on the Maxima\_Setup file. Follow the prompts to install the software. Different versions will be installed. We will always use the version called wxMaxima. Be careful to choose the correct one. You will find a general introduction to maxima in the Integrating ICT and Maths module. However, there is a complete manual for the software available. To find it, run wxMaxima and choose *Maxima help* in the *Help* menu. The web site for this software is <http://maxima.sourceforge.net>. Look in activity 3 to see how to get started using mxMaxima for matrix operations.



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### **XIII. Useful links**

#### **Wolfram Mathworld (Visited 07.11.06)**

<http://mathworld.wolfram.com/>

Wolfram Mathworld is an extremely comprehensive encyclopedia of mathematics.

This link takes you to the home page. Because Calculus is such a wide topic, we recommend that you search Mathworld for any technical mathematical words you find. For example, start by doing a search for the word 'limit'.

#### **Wikipedia (visited 07.11.06)**

[http://en.wikipedia.org/wiki/Main\\_Page](http://en.wikipedia.org/wiki/Main_Page)

Wikipedia is a general encyclopedia. However its mathematics sections are extremely good. If you don't find what you want at Mathworld, try searching at Wikipedia. It is often good to check both, to see which is easier to understand.

#### **MacTutor History of Mathematics (visited 07.11.06)**

[http://www-history.mcs.st-andrews.ac.uk/HistTopics/The\\_rise\\_of\\_calculus.html](http://www-history.mcs.st-andrews.ac.uk/HistTopics/The_rise_of_calculus.html)

html

Read for interest, the history of calculus.

#### **Asguru (Visited 07.11.06)**

<http://www.bbc.co.uk/education/asguru/maths/12methods/04integration/index.shtml>

shtml

This link gives access to a number of pages which introduce integration. Click the links in turn to work through the different sections. When you have read to the

bottom of a page click the large right arrow. This will take you to the next page in the section. (Save the definite integral section for activity 2 later in this unit).

Many of the sections contain interactive activities. Read the instructions carefully and explore the ideas.

Some sections contain interactive tests and exercises. Use these to check your understanding.



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**Nrich (Visited 07.11.06)**

<http://www.nrich.maths.org/public/leg.php?code=-211>

Try these puzzles on the theme of integration.

**Wolfram Mathworld (Visited 07.11.06)**

<http://mathworld.wolfram.com/>

Search for: integral, integrate, anti-derivative. [If you are using this document on a computer, then the links can be clicked directly].

**Asguru (Visited 07.11.06)**

<http://www.bbc.co.uk/education/asguru/maths/13pure/03sequences/index.shtml>

This link gives access to a number of pages which introduce sequences and series.

Click the links in turn to work through the different sections. When you have

read to the bottom of a page click the large right arrow. This will take you to the next page in the section.

Many of the sections contain interactive activities. Read the instructions carefully and explore the ideas.

Some sections contain interactive tests and exercises. Use these to check your understanding.

### **Nrich (Visited 07.11.06)**

<http://www.nrich.maths.org/public/index.php>

Type sequences and series into the search box in the top right hand corner of the page.

Try these sequences and series puzzles.

There are a very large number to choose from. We recommend you only look at stage 4 and 5 puzzles. Try the first one(s) that look interesting.

### **Wolfram Mathworld (Visited 07.11.06)**

<http://mathworld.wolfram.com/Series.html>

Read this entry for Series.

Follow links to explain specific concepts as you need to.



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### **Wikipedia (visited 07.11.06)**

[http://en.wikipedia.org/wiki/Sequence\\_and\\_series](http://en.wikipedia.org/wiki/Sequence_and_series)

Read this entry for Sequences and Series.

Follow links to explain specific concepts as you need to.

### **Wolfram Mathworld (Visited 07.11.06)**

<http://mathworld.wolfram.com/PartialDerivative.html>

Read this entry for Partial Derivatives.

Follow links to explain specific concepts as you need to.

**Wikipedia (visited 07.11.06)**

[http://en.wikipedia.org/wiki/Partial\\_derivative](http://en.wikipedia.org/wiki/Partial_derivative)

Read this entry for Partial Derivatives.

Follow links to explain specific concepts as you need to.



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#### **XIV. learning activities**

### **Mathematics Module 3: Calculus**

#### **Unit 1: Elementary Differential Calculus (35 hrs)**

##### **TITLE: Limits and Continuity (6 Hours)**

##### **Specific Learning Objectives**

At the end of this activity the learner should be able to:

- Determine one-sided limits;
- Determine whether or not a limit exists;
- Evaluate limits;
- Determine continuity of a function at a point and over an interval.

##### **Summary**

The limit concept is at the very heart of mathematical analysis. The concept is crucial in defining the related key mathematical concepts of continuity, differentiability and integrability of functions of a single variable.

With the aid of three relevant real life stories in an African context, we shall introduce the concepts of limit and continuity. Using the stories we shall engage you, the learner, with the twin ideas and thereby enable you acquire knowledge



of and competence in not only evaluating limits and determining continuity of functions but also assist in laying exposing the challenges you are likely to face in an actual classroom situation.

## Compulsory Reading

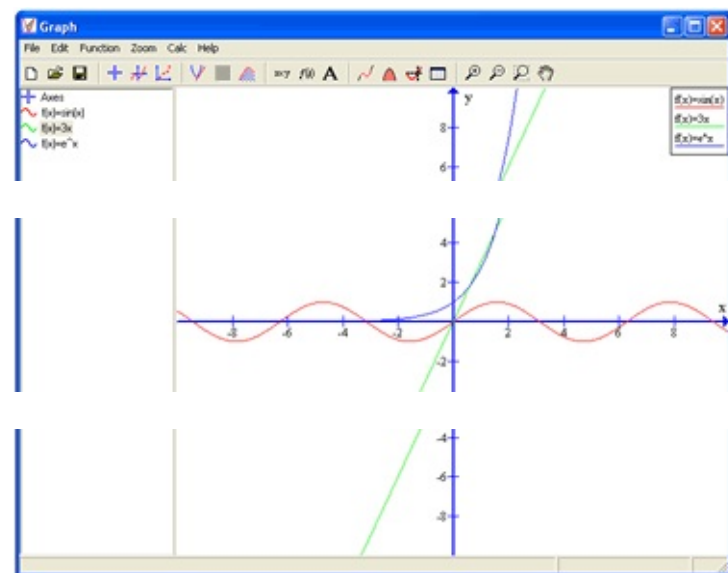
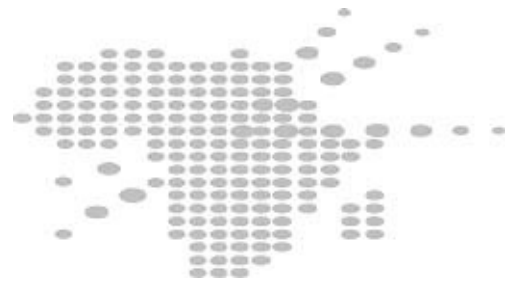
All of the readings for the module come from Open Source text books. This means that the authors have made them available for any to use them without charge.

We have provided complete copies of these texts on the CD accompanying this course.

**The Calculus Bible, Prof. G.S. Gill:** Brigham Young University, Maths Department.

Brigham Young University – USA.

Limits and Continuity, Chapter 2, pp 35 – 94.



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**Internet and Software Resources**

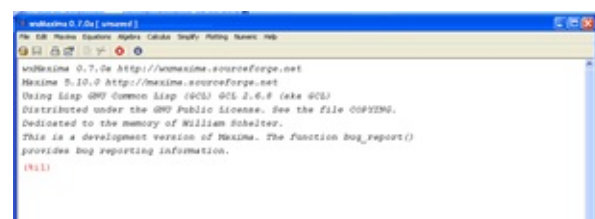
For the Calculus course we have provided copies of two pieces of open source software. You are free to use this software without charge. You should install the software and make sure you have access to a computer in order to use them. In the first case the software provides easy to use graphing tools and in the second case it provides open tools to explore mathematics in general. You should use this software as often as possible, so that you get used to how it works.

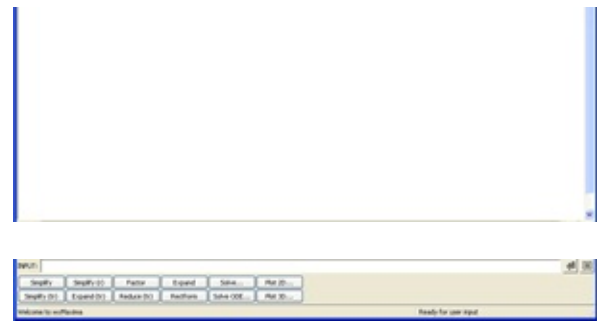
**Graph.** This is an easy to use graph drawing program. Whenever you need to see the graph of a function, you should use graph.

- a. Install the software by double clicking on *SetupGraph*.
- b. Run the program by double clicking on the **Graph** icon.
- c. Click CLOSE to get past the tip-of-the-day.
- d. Press the insert key on your keyboard.
- e. Click in the box labeled  $f(x)=$
- f. Type in a function and click OK.
- g. Double click on the function to change the graph properties.
- h. You should spend a little time exploring the software. Try out all of the menus and experiment with the features.

The Graph software showing 3 graphs

**The Graph software showing 3 graphs**





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**wxMaxima.** This is Computer Algebra System (CAS). You should double click on the Maxima\_Setup file. Follow the prompts to install the software. Different versions will be installed. We will always use the version called wxMaxima. Be careful to choose the correct one. You will find a general introduction to maxima in the Integrating ICT and Maths module. However, there is a complete manual for the software available. To find it, run wxMaxima and choose *Maxima help* in the *Help* menu. The web site for this software is <http://maxima.sourceforge.net>. Look in activity 3 to see how to get started using mxMaxima for matrix operations.

### Getting Started with wxMaxima

- Launch **wxMaxima**
- Your screen should look like this:

Type your

commands

in here

Be careful when typing :

- 

Don't add extra spaces or

punctuation.

- 

Make sure you choose the

correct brackets.

- 

When you open a bracket, the  
close bracket is automatically

↙

entered.

- You type mathematical commands, then press the RETURN key on your keyboard.
- Type  $x^3$  then press RETURN. This is how to type  $x^3$ .
- Now type  $2*x+3*x$  and press RETURN. Notice that **wxMaxima** automatically simplifies this to  $5x$ .
- Look at the **wxMaxima** manual (choose *Maxima help* in the *Help* menu).
- Find new commands to try out.
- Spend time practicing. **wxMaxima** is quite difficult to get started with, so it is important to practice.

**Note:** You **must not** use graph or wxMaxima to answer exercise questions for you! Instead, you should try different examples of calculations and operations to make sure that you understand how they are done, so that you are better able to do them **without** the support of the software.



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## Web References

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html

Read for interest, the history of calculus.

### **Key Concepts**

#### **Limit of a function**

A function  $f(x)$  has a limit  $L$  as  $x$  approaches point  $c$  if the value of  $f(x)$

$\lim$

approaches  $L$  as  $x$  approaches  $c$ , on both sides of  $c$ . We write

$$f(x) = L$$

$x$

.

$\rightarrow c$

#### **One sided limits**

A function may have different limits depending from which side one approaches  $c$ . The limit obtained by approaching  $c$  from the right (values greater than  $c$ ) is called the right-handed limit. The limit obtained by approaching  $c$  from the left (values less than  $c$ ) is called the left-handed limit. We denote such limits by

$\lim$

$\lim$

+

$$f(x) = L,$$

–

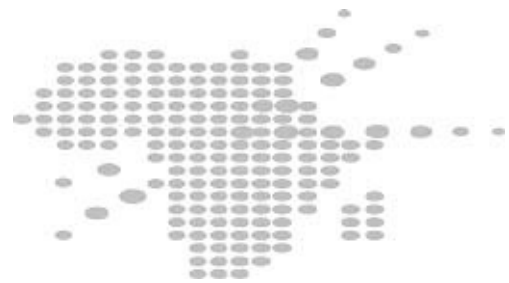
$$f(x) = L$$

+

$$x \rightarrow c$$

–

$$x \rightarrow c$$



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## Continuity of a function

A function  $f(x)$  is said to be continuous at point  $c$  if the following three conditions are satisfied:

- The function is defined at the point, meaning that  $f(c)$  exists,
- The function has a limit as  $x$  approaches  $c$ , and
- The limit is equal to the value of the function. Thus for continuity we must

lim

have

$$f(x) = f(c).$$

$$x \rightarrow c$$

## Continuity over an Interval

A function  $f(x)$  is continuous over an interval  $I = [a, b]$  if it is continuous at each interior point of  $I$  and is continuous on the right hand at  $x = a$  and on the

left hand at  $x = b$ .

$b$

## Discontinuity

A function has a discontinuity at  $x = c$  if it is not continuous at  $x = c$ . For exam-

$\sin x$

ple,  $f(x) =$

has a discontinuity at  $x = 0$ .

$x$

## Jump discontinuity

A function  $f$  is said to have a jump discontinuity at  $x = c$  if

$$f(x) = M$$

$\lim$

,

$x \rightarrow c$

Ⓜ

$$f(x) = L$$

$\lim$

, but  $L \neq M$ .

$x \rightarrow c^+$

Ⓜ

## Removable discontinuity

A function  $f(x)$  has a removable discontinuity at  $x = c$  if either  $f(c)$  does not

exist or  $f(c) \neq L$

## Partial derivative

A partial derivative of a function of several variables is the derivative of the

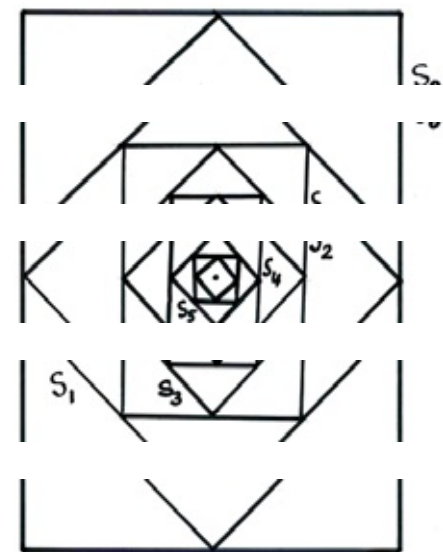
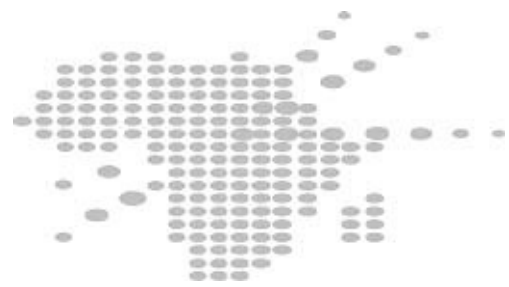
given function with respect to one of the several independent variables, treating



all the other independent variables as if they were real constants. For, example,

if  $f(x,y,z) = x^2 y + 3xz^2 - xyz$  is a function of the three independent variables,  $x$ ,  $y$ , and  $z$ , then the partial derivative of  $f$  with respect to the variable  $x$  is the

function  $g(x,y,z) = 2xy + 3z^2 - yz$ .



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## Key Theorems and/or Principles

### Intermediate Value Theorem for Continuous Functions

If  $f$  is continuous on  $[a, b]$  and  $k$  is any number lying between  $f(a)$  and  $f(b)$  then there is at least one number  $c$  in  $(a, b)$  such that  $f(c) = k$ .

### Continuity of a differentiable function

If  $f$  is differentiable at  $c$  then  $f$  is continuous at  $c$ .

$c$

## Learning Activity

### Engaging Stories

#### (a) Story of shrinking areas

Consider the geometric pattern resulting  
from a process of inscribing squares  
inside a given square in the following  
manner:

Start with a unit square  $S_0$ . In  $S_0$  inscribe

0

0

a square  $S_1$  whose vertices are the mid-

1

points of the sides of  $S_0$ . In  $S_1$  inscribe

0

1

a square  $S_2$  whose vertices are the mid-

2

points of the sides of  $S_1$ . Carry out this process a number of times to obtain the

1

squares  $S_k$ ,  $k$

.

k

=,

1

,

3

,

2 4

# Questions

1. How far can one go in inscribing such squares?
2. What happens to the areas of the inscribed squares  $S_n$  as the inscription process is continued?

3. What are the areas of the squares  $S_k$ ,

?

$k$

$=$ ,

1

,

3

,

24





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### **(b) Story of evaporating water**

Francis is a nurse at the village health centre that is close to his house. One afternoon during lunch break Francis decided to dash home and prepare himself a cup of tea. He went to the kitchen, poured some water into a cooking pot, lighted the kerosene stove and placed the pot on to boil the water.

While waiting for the water to boil, Francis quietly settled in a couch, opened his radio and listened to the latest news broadcast. Unfortunately, he got carried away by the live news being aired on the war that was being fought in Lebanon and forgot all about the water that was boiling in the kitchen.

### **Questions**

1. What do you think was happening to the volume of the boiling water as time passed by?
2. What will happen to the volume of the water in the pot if Francis forgets all about what he came back home for?

### **(c) Story of a washed away bridge**

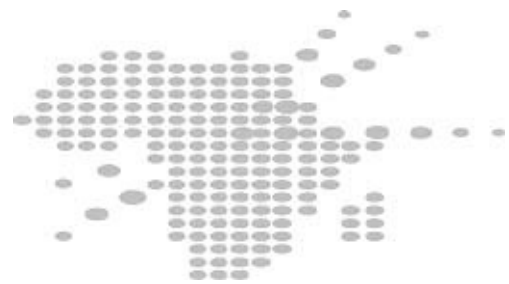
Jesika is Headmistress of a school which lies a short distance but on the other side of a river that separates her village from

the school. To go to school Jesika walks along a road which goes over a bridge that joins the two villages.

One morning Jesika could not reach her school. The bridge had been washed away by floods caused by a heavy downpour of rain that occurred the night before.

### Questions

1. Was the road from Jesika's home to school continuous after the bridge had been washed away?
2. How could one make the road from Jesika's home to school once again passable?



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**The first two stories** in the introduction relate to the concept of **limit of a function**. Referring to the questions raised at the end of the first story you will most likely have arrived at the following correct answers:

**1. Answer to Question 1:** The process of inscribing the squares is unending.

However, after only a few steps one is forced to stop the process because the sides of the squares become too small to allow further visible bisection.

**2. Answer to Question 2:** The areas of the inscribed squares get smaller and smaller as  $n$  increases.

**3. Answer to Question 3:** If we denote the area of square  $S$  by  $A$  then one

$k$

$k$

finds  $A = 2^{-1}$ ,

$A = 2^{-3}$ ,

$A = 2^{-5}$ ,

$A = 2^{-7}$ .

1

2

3

4

In general, one finds the area of the square  $S$  to be

$A = 2^{-(2n-1)}$ .

$n$

$n$

These answers, and especially the answer to the third question, reveal that as  $n$  gets very large the areas of the squares get very small, ultimately shrinking to zero.

The same experience is made from the answers to questions raised in connection with the story of the boiling water. As time passes by, the water evaporates and hence its volume in the pot decreases. If left unattended, the water will evaporate completely and the volume of water left in the pot will be zero and thereby creating a high risk of causing fire in the house!

**The third story** relates more to the concept of continuity of a function. The road

from Jesika's home to the school represents the curve of a function. When the bridge is in place the road is in one piece. It is continuous. When the bridge is washed away by the flood waters the road is disconnected. The function is then not continuous. To make it once again continuous one must construct the bridge. In this case the discontinuity is removable.



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## Mathematical Problems

### Determination of Limits

Without leaning on relevant stories such as the ones given above, the limit concept is pedagogically difficult to introduce starting directly with a mathematical example. However, now that we have broken the ice, we are ready to confront directly a mathematical problem.

2

$x - 4$

**Problem:** Find the **one-sided limits** of the function  $y =$

as  $x$  tends to 2.

$x - 2$

Does the function have a limit at 2? Determine whether or not the function is continuous at  $x = 2$ . If it is not continuous discuss whether or not the discontinuity is removable.

Note that the function is not defined at 2. However, this should not bother us much because "**tending to 2**" does not require being at the point itself. Indeed,

this is a very important observation to make regarding limits. The function need not be defined at the point. In order to solve the problem let us investigate the behavior of the function as  $x$  approaches 2 by evaluating it at values close to 2 on both sides of 2:

$x$

$$y = f(x)$$

1.5

3.5

1.7

3.7

1.9

3.9

1.95

3.95

1.97

3.97

1.99

3.99

2.0

4!

2.01

4.01

2.03

4.03

2.05



4.05  
2.1  
4.1  
2.3  
4.3  
2.5  
4.5

**DO THIS:** Draw the graph of the function.

How do you denote that the number 2 is not in the domain of the function?



African Virtual University 0

2

46–

y 2

$$= x^2 - 4$$

1

23– 1

23 x – 2

From the values given in the table, it is quite obvious that, as  $x$  approaches 2 from either side, the function values approach 4 . In this case the **left-hand** and **right-hand** limits are equal.

y

6

4

2

x

-3

-2

-1

1

2

3

-2

$$y = x^2 - 4$$

$$x - 2$$

### Software Task

Reproduce the graph using the software program Graph.

You will need to type into the f(x)= box:  $(x^2-4)/(x-2)$

Experiment with other similar functions.

How can we explain this somewhat surprising result?

Examine the function more closely. Note that the expression  $x^2 - 4$  may be

$x^2 - 4$  may be

expressed as the product of two factors  $(x + 2)(x - 2)$ . We can therefore write

$$(x + 2)($$

$$x - 2)$$

2

the function in the form

$$f(x) =$$

$$x - 2$$

Since  $x \neq 2$  we can cancel out the factor  $x - 2$  and get the linear equation

$y = x + 2$  valid everywhere away from the point  $x = 2$ . Sketch the graph of

$y = f(x)$ . As  $x$  approaches 2 taking values less than 2 or values greater than 2

the function values approach 4.



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### Group Work

(i) Get together with a colleague or two and discuss the proofs and significance of the theorems on limits of:

- A constant function

$$f(x) = k$$

- Sum of two functions

$$f(x) + g(x)$$

- Difference of two functions

$$f(x) - g(x)$$

- Product of two functions

$$f(x) \cdot g(x)$$

$$f(x) / g(x)$$

- Quotient of two functions  $g(x)$

(ii) Study carefully how to compute limits at infinity (**involving**  $x \rightarrow \infty$ ) by

working out the examples given in pp 96-98.

## Determination of Continuity

We have used the above mathematical example to demonstrate the concept of a limit. Fortunately, the same example can be used to illustrate the concept of continuity.

Because the function is **not defined at**  $x = 2$  we can immediately conclude that it is **not continuous** at  $x = 2$ .

However, since the limit exists, the function has what is known as a **removable discontinuity**. One removes such a discontinuity by defining a new function  $F(x)$  that is identical to the given function  $f(x)$  away from the point of discontinuity, and has the value of the limit of  $f(x)$  at the point of discontinuity.

$$f(x) = \begin{cases} x - 4 & | \ x \neq 2 \\ \text{undefined} & | \ x = 2 \end{cases}$$

The function

$$F(x) = \begin{cases} x - 4 & | \ x \neq 2 \\ 4 & | \ x = 2 \end{cases}$$

is therefore continuous at  $x = 2$ .

$x = 2$



African Virtual University

### Group Work

- (i) Get together with a colleague or two and discuss the properties of continuous functions, related theorems and their proofs as presented in pp 84-94.

Use the **Graph** software to experiment with different functions  $f(x)$  and  $g(x)$

Look at the graphs of  $f(x) + g(x)$ ,  $f(x) - g(x)$ ,  $f(x)g(x)$ ,

$g(x)$

Write a short commentary on what you notice.

(ii) Take special note to the **Intermediate Value Theorem** and its applications

(pp page 86) significance of the theorems on limits of:

- A constant function

$$f(x) = k$$

- Sum of two functions  $f(x) + g(x)$

- Difference of two functions  $f(x) - g(x)$

- Product of two functions  $f(x)g(x)$

$f(x)$

- Quotient of two functions  $g(x)$

as given and proved in pp 44 – 48.

(iii) Study carefully how to compute limits at infinity (**involving  $x^{\infty}$** ) by

working out the examples given in pp 96-98.

**Exercise Questions (from Gill: The Calculus Bible)**

**DO THIS**

Solve all the problems given in:

**Exercise 2.1(pp 60-61)**

**Exercise 2.2 (pp 71-72)**



## **Unit 1: Elementary Differential Calculus (35 Hrs)**

### **Learning Activity # 2**

#### **TITLE: Differentiation of functions of a single variable (29 Hrs)**

#### **Specific Learning Objectives**

At the end of this activity the learner should be able to:

- Determine the differentiability of a function of a single variable.
- Differentiate a given function, including: polynomial, rational, implicit and inverse functions.
- Find higher order derivatives.
- Apply derivatives to:
  - (i) Evaluate certain limits of functions (L'Hopitals Rule),
  - (ii) Calculate relative extreme values of a function (Second Derivative Test) and
  - (iii) Obtain a series expansion of a function at a point (Taylor's series).

#### **Summary**

The derivative of functions of a single variable is a concept that is built on the foundations already laid down by the concepts of limit and continuity. Continuity measures zero in order of smoothness of a curve while differentiability is of order one. You will soon discover that differentiability is a sufficient condition for continuity but continuity is only a necessary condition for differentiability.

In this learning activity the derivative of a function will be defined. Rules for differentiation will be stated. Derivatives of some special mathematical functions will be derived. Key theorems will be stated and demonstrated and some useful mathematical applications of derivatives will be stated and demonstrated.

## Compulsory Reading

### The Calculus Bible

**Prof. G.S. Gill:** Brigham Young University, Maths Department.

Differentiation,

Chapter 3, pp 99 – 137

Applications,

Chapter 4, Section 4.1, pp 146 – 179.



African Virtual University 44

### Internet and Software Resources

#### Software

You will be asked to complete tasks using **wxMaxima** in this activity.

#### Web References

##### **NRich (Visited 07.11.06)**

[http://www.nrich.maths.org/public/viewer.php?obj\\_id=477&part=index&refpage=monthindex.php](http://www.nrich.maths.org/public/viewer.php?obj_id=477&part=index&refpage=monthindex.php)

Read this interactive introduction to differentiation.

##### **ASGuru (Visited 07.11.06)**

<http://www.bbc.co.uk/education/asguru/maths/12methods/03differentiation/index.shtml>

This link gives access to a number of pages which introduce differentiation. Click the links in turn to work through the different sections. When you have read to the bottom of a page click the large right arrow. This will take you to the next

page in the section.

Many of the sections contain interactive activities. Read the instructions carefully and explore the ideas.

Some sections contain interactive tests and exercises. Use these to check your understanding.

### **Wolfram Mathworld (Visited 07.11.06)**

<http://mathworld.wolfram.com/>

Search for: differentiation, derivative, L'Hospitals Rule, Second Derivative Test, Taylor's series. [If you are using this document on a computer, then the links can be clicked directly].

### **Wikipedia (visited 07.11.06)**

[http://en.wikipedia.org/wiki/Main\\_Page](http://en.wikipedia.org/wiki/Main_Page)



African Virtual University 45

## **Key Concepts**

### **Critical points**

A value  $x = c$  in the domain of a function  $f(x)$  is called a critical point if the derivative of  $f$  at the point is either zero or is not defined.

### **Critical values**

The value of a function  $f(x)$  at a critical point of  $f$  is called a critical value.

### **Derivative of a function**

$$\left[ \frac{f(x) - f(c)}{x - c} \right]$$

The derivative of a function  $y = f(x)$  is defined either as  $\lim$



]

|

|

$x \rightarrow c$  [

$x - c$

]

[  $f(c + )$

$h - f(c)$

or as  $\lim$

]

|

| provided that

$h \rightarrow$

[

$h$

$0$

]

the limit exists.

**Differentiable**

A function  $f(x)$  is said to be differentiable at  $x = c$  if

[  $f(x) - f(c)$

$\lim$

] exists.

$x \rightarrow c$

$x - c$

## Implicitly defined functions

A function  $y$  is said to be implicitly defined as a function of  $x$  if  $y$  is not isolated on one side of the equation. For example, the equation

2

2

$xy - 2$

3

$x y + x = 3$

defines  $y$  implicitly as a function of  $x$ .

## Implicit differentiation

Implicit differentiation is a method of differentiating a function that is defined implicitly, without having to solve the original equation for  $y$  in terms of  $x$ .

## Necessary condition

$P$  is said to be a necessary condition for  $Q$  if whenever  $Q$  is true then  $P$  is also true. For example, continuity ( $P$ ) is a necessary condition for differentiability ( $Q$ ).

In brief one says:  $P$  is implied in  $Q$ , written as  $P \leftarrow Q$



African Virtual University 46

## Necessary and sufficient condition

$P$  is said to be a necessary and sufficient condition for  $Q$  if  $P$  implies  $Q$  and  $Q$  implies  $P$ . For example, If a triangle is equilateral ( $P$ ), then its three angles are equal ( $Q$ ), and if the three angles of a triangle are equal ( $Q$ ) then the triangle is equilateral ( $P$ ). One writes  $P \leftrightarrow Q$

## Slope of Tangent

If a function  $y = f(x)$  is differentiable at  $x = c$  then the slope of the tangent to the graph of  $f$  at the point  $(c, f(c))$  is

$f'(c)$ .

## Sufficient condition

$P$  is said to be a sufficient condition for  $Q$  if whenever  $P$  is true then  $Q$  is also true. In other words,  $P$  implies  $Q$ . For example, If a triangle is equilateral, then it is isosceles. One writes  $P \rightarrow Q$

## Key Theorems and/or Principles

### Rolle's Theorem

If  $f$  is continuous on  $[a, b]$

and differentiable on  $(a, b)$

with  $f(a) = f(b)$

then,

there exists at least one number  $c \in (a, b)$

such with

$f'(c) = 0$

### Mean Value Theorem of Differentiation (MVT)

If  $f$  is continuous on  $[a, b]$

and differentiable on  $(a, b)$

then there exists at

least

one number  $c \in (a, b)$

such with  $f'(c) = \frac{f(b) - f(a)}{b - a}$

.

$b - a$

**N.B:** 1. Rolle's Theorem is a special case of the Mean Value Theorem.

2. The MVT roughly asserts that for any differentiable function on an interval

$f(b) - f(a)$

$(a, b)$

the derivative  $f'(x)$  takes on its mean value

somewhere

$b - a$

inside the interval.



African Virtual University 47

### Cauchy's Mean Value Theorem

If  $f$  and  $g$  are both continuous on  $[a, b]$

and differentiable on  $(a, b)$

and if

$g'(x) \neq 0$   $\forall x \in (a, b)$

$\forall x \in (a, b)$

then there exists at least one number  $c \in (a, b)$

such with

$f(b) - f(a)$

$= (b - a) f'(c)$

$f'(c)$

$=$

$g()$

$b - g(a)$

$g'(c)$

#### 4. Taylor's Theorem

Let  $f$  be a function whose  $n$  derivatives are continuous on the closed inter-

val  $[c, c + h]$  (or  $[c + ,$

$h c]$  if  $h$  is negative) and assume that  $(n+ )1$

$f$

exists in

$[c, c + h]$  (or  $[c + ,$

$h c]$  if  $h$  is negative). Then, there exists a number  $\theta$ , with

$0 < \theta < 1$  such that

$n h k$

$f(c + )$

$(k)$

$h =$

$f$

$(c) +$

$(,$

$h \theta)$

$\Sigma$

$R_{n+}$

$k=0$

!

1

k

n 1

h +

Where

(c,  $\theta$ )

(n+ )

1

=

f

(c +

)

h

R

$\theta$

n 1

+

(n + )!

1





African Virtual University 48

## Learning Activity

### Story of swinging a stone tied to a string

Gisela was very fond of playing using strings. One day she tied a stone at the end of a string, held the other end in her right arm and began to swing the stone rapidly in circles.

### Question

If Gisela lets go of the string, which direction would the stone fly off?

### Mathematical Problem

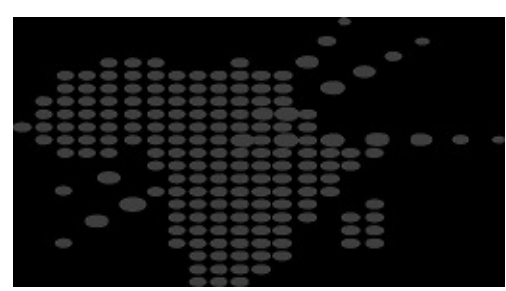
In responding to the question raised at the end of the stone-swinging story, some learners, and indeed many people are inclined to think that the stone would continue on a curved path. However, this is far from the truth. Newton's Law of motion dictates that the stone will fly off on a straight line **tangential to the circle** at the point the string is let go.

The concept of a derivative is best introduced by discussing the slope of a **tangent to a curve**. (See photo above)

For a small value of the interval  $h > 0$  let  $Q(x + ,$

$h f(x + h)$  be a point on

the curve representing the function  $y = f(x)$  that is close to another point  $P(x, f(x))$  on the same curve. (See photo above)



African Virtual University 49

The **slope of the chord** joining the two points is given by the quantity

PQ

$f(x+h) - f(x)$  that is the quotient of the increase in  $f$  over the increase in

$h$

$x$ . We now allow point  $Q$  slide down the curve towards point  $P$ . This implies

the distance  $h$  separating the two points getting smaller. In that process, the

chord rotates towards a limiting position. This limiting chord is the

PQ

tangent to the curve at  $P$ . Thus the slope of the tangent at  $P$  is given by

$f(x+h) - f(x)$

lim

.

$h \rightarrow 0$

$h$

This limit may exist or may not exist. If not, then there is no tangent to the curve

at that point, or the tangent has an infinite slope.

### Group Work

(i) Get together with a colleague or two and discuss the rules for differentiating

sums, products and quotients of functions and their respective proofs as presen-



ted in pp 101-107. Take special note of the **Chain Rule** and its applications (pp 111-116).

(ii) Read the contents of the following sections and do the exercises at the end of each:

- Section 3.3: Differentiation of inverse functions (pp 118-129)

Do Exercise 3.3 (pp 129-130)

- Section 3.4: Implicit differentiation (pp 130-137)

Do Exercise 3.4 (pp 136-137)

- Section 3.5: Higher order derivatives (pp 137-155)

Do Exercise 3.5 (pp 144-145)

- Section 4.1 of Chapter 4: Applications of differentiation (pp 146-157),

paying special attention to the theorem on Relative maxima and minima (page 147) and L'Hopital's Rule (page 150).

Do Exercise 4.1 (pp 156-157)



African Virtual University 50

### Software Task

- Launch wxMaxima
- Experiment with differentiating different functions.

Example:

Type `diff(x^3,x)`.

This asks **wxMaxima** to differentiate  $x^3$  with respect to  $x$ .

Now press RETURN.

You should see the derivative which is  $3x^2$ .

Now try this one:  $\text{diff}(x^4+3*x^2+2,x)$

- Work out methods for differentiating different families of functions.
- Write a report describing your methods.
- Check the results you obtained in your group work task. Make up your own examples in each case and check them using **wxMaxima**.

### **Exercise Questions (From Gill: The Calculus Bible)**

DO THIS

Solve all the problems given in

**Exercise 3.1(pp 108-111)**

**Exercise 2.2 on page 116**



African Virtual University 51

### **Unit 2: Elementary integral Calculus (35 Hrs)**

#### **Learning Activity No.1**

**TITLE:Anti-differentiation, indefinite integrals and Methods of integration (14 Hrs)**

#### **Specific Learning Objectives**

At the end of this activity the learner will be able to:

- Find anti-derivatives of standard mathematical functions
- Apply various methods of integration

#### **Summary**

In the second learning activity of Unit 1 you saw how differentiation of a function

$f'(x)$  led to a new function'

$f'(x)$  called the derivative of  $f$ . Now we discuss

another possibility of deriving a function  $F(x)$  from  $f$ . We denote the new

function by  $F(x) = \int f(x) dx$

∫

This function is referred to as an **indefinite integral**, **anti-derivative** or a **primitive of  $f$** .

An expression for the function  $F(x)$  is obtained by requiring that its derivative is  $f(x)$ .

$$F'(x) = f(x)$$

The process of finding an anti-derivative is referred to as **anti-differentiation** or integration. It is the inverse of differentiation.

In the current activity you will learn rules and apply various methods of integration on a variety of functions, including polynomial, exponential, logarithmic and trigonometric functions.



African Virtual University 52

### **Compulsory Reading**

All of the readings for the module come from Open Source text books. This means that the authors have made them available for any to use them without charge.

We have provided complete copies of these texts on the CD accompanying this course.

**The Calculus Bible, Prof. G.S. Gill:** Brigham Young University, Maths Department.

Brigham Young University – USA.

Chapter 6: Techniques of integration (pp 267-291)

## **internet and Software Resources**

### **Software**

Use **wxMaxima** to check your solution to integration problems.

You should use the **integrate** function. It works like this:

- $\text{Integrate}(x^2,x)$  will integrate the function  $x^2$  with respect to  $x$ .
- $\text{Integrate}(x^2,x,0,2)$  will complete a definite integration of  $x^2$  with respect to  $x$  with limits from  $x = 0$  to  $x = 2$ .

Look for more advanced functions in the **wxMaxima** manual.

### **Web References**

#### **Asguru (Visited 07.11.06)**

<http://www.bbc.co.uk/education/asguru/maths/12methods/04integration/index.shtml>

This link gives access to a number of pages which introduce integration. Click the links in turn to work through the different sections. When you have read to the bottom of a page click the large right arrow. This will take you to the next page in the section. (Save the definite integral section for activity 2 later in this unit).

Many of the sections contain interactive activities. Read the instructions carefully and explore the ideas.

Some sections contain interactive tests and exercises. Use these to check your understanding.



African Virtual University 53

### **Nrich (Visited 07.11.06)**

<http://www.nrich.maths.org/public/leg.php?code=-211>

Try these puzzles on the theme of integration.

### **Wolfram Mathworld (Visited 07.11.06)**

<http://mathworld.wolfram.com/>

Search for: integral, integrate, anti-derivative. [If you are using this document on a computer, then the links can be clicked directly].

### **Wikipedia (visited 07.11.06)**

[http://en.wikipedia.org/wiki/Main\\_Page](http://en.wikipedia.org/wiki/Main_Page)

## **Key Concepts**

### **Anti-derivatives**

Anti-derivatives are a family of functions  $\{ F(x) + c \}$

$c$ , where  $c$  is an arbitrary

constant of integration, with a common derivative  $f(x)$ .

### **Anti-differentiation**

Anti-differentiation is the process of going from a derivative function  $f(x)$  to a function  $F(x)$  that has that derivative.

### **Indefinite integral**

An anti-derivative is also referred to as an indefinite integral due to the presence of the arbitrary constant of integration.

### **Primitive function**

If  $f$  is a function defined on an interval  $I$ , then the function  $F$  is a primitive of  $f$  on  $I$  if  $F$  is differentiable on  $I$  and  $F'(x) = f(x)$ .



African Virtual University 54

## Key Theorems and/or Principles

### Rules for Integration

1.

$$\int a f(x) dx = a \int f(x) dx$$

$\int$

$\int$

$n$

$n$

$($

$)$

2.

$$\int a f(x) dx =$$

$a$

$$\int f(x) dx$$

$$\int \sum_{i=1}^n f_i(x) dx =$$

$$\sum_{i=1}^n \int f_i(x) dx$$

$i$

$\int$

$)$

$\int_{i=1}^n$ 

)

 $i=1$  $n+1$  $x$ 

3.

 $n$  $x dx =$  $+ C,$  $\int$  $n \neq -1$  $n+1$ 

4.

 $-1$  $x dx = \ln x + C$  $\int$ 

(G.S. Gill Section 6.1 pp 267)

### Learning Activity

There are two ways of introducing the concept of integration. Historically the concept of integration was introduced via the problem of finding the area below a curve lying between two given ordinates. This is the Riemann approach that leads to a sequence of Riemann Sums that under some fairly simple assumptions on the function  $f$  can be shown to converge to the number (area).

The other way of introducing the integral of a function is by viewing integration as an inverse operation to differentiation. This led to calling the process **anti-**

**differentiation** and the outcome of the process being called an **anti-derivative**

or a **primitive** of the function  $f$ .

**Definition:** A function  $F(x)$  is called a primitive (or anti-derivative) of another function  $f(x)$  if the derivative of  $F(x)$  is  $f(x)$ :

$$F'(x) = f(x).$$

One denotes  $F(x)$  by  $\int f(x) dx$

$\int$

. The symbol  $\int$  is the integral sign,

and the presence of  $dx$  is a reminder of the inverse process of differentiation (anti-differentiation).



African Virtual University 55

### Examples

#### FuNCTIoN

#### Pr i m i t i v e

$$f(x)$$

$$F(x) + C$$

$$2$$

$$x$$

$$13$$

$$x + C$$

$$3$$

$$\cos x$$



$$\sin x + C$$

$$\sin x$$

$$-\cos x + C$$

$$x$$

$$e$$

$$x$$

$$e + C$$

$$1$$

$$\ln x + C$$

$$x$$

Note that the function

$G(x) = F(x) + C$ , where  $C$  is an arbitrary real

number, is also a primitive of  $f(x)$  because  $G'(x) = F'(x)$ .

This implies that a primitive of a function is not unique, however, any two different primitive functions of a given function differ only by a constant:

$$F(x) - G(x) = C$$

### **Rule Governing Primitive Functions**

If  $F, G$  are primitive functions of  $f, g$  on an interval  $I$  then  $F \pm G$  is a primitive of  $f \pm g$ .

### **Do THIS**

Find the primitive functions of the following functions

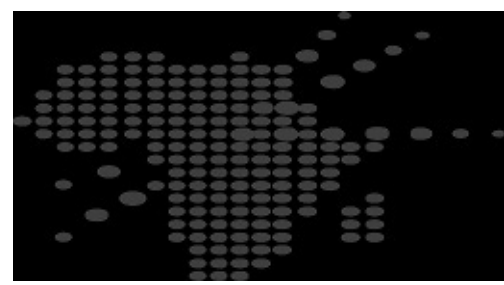
1.

2

$$f(x) = 3x + 2x + 1$$

2.  $f(x) = \tan x$

3.  $f(x) = x + 4$



African Virtual University 56

## **Methods of integration**

**G. S. Gill (Chapter 6)** lists a variety of methods for integrating various types of

functions. These include:

Integration by substitution

-

Section 6.2 (page 272)

Integration by parts

-

Section 6.3 (page 276)

Trigonometric integrals

-

Section 6.4 (page 280)

Trigonometric substitution

-

Section 6.5 (page 285)

Use of Partial Fractions

-

Section 6.6 (page 288)

Numerical Integration

-

Section 6.9 (page 292)

## **Do THIS**

Look up the various methods presented in the above Sections of your Self- Instructional Material and ensure you master them. Discuss them with some colleagues and test yourselves by solving as many problems as you can in the exercises listed in the following section.

### **Exercises (From Gill: The Calculus Bible)**

Solve the problems in the following Exercises:

Exercise 6.1 (page 272)

Exercise 6.2 (page 273)

Exercise 6.3 (page 276)

Exercise 6.4 (page 282)

Exercise 6.5 (page 285)

Exercise 6.6 (page 288)

Exercise 6.9 (page 293)



African Virtual University 57

## **Unit 2 Elementary integral Calculus (35 Hrs)**

### **Learning Activity No.2**

**TITLE: Definite integrals, improper integrals and Applications (21 Hrs)**

#### **Specific Learning Objectives**

At the end of this activity the learner will be able to:

- Evaluate definite integrals
- Apply numerical methods in evaluating definite integrals

- Evaluate convergent improper integrals
- Apply knowledge of integration in solving some mathematical problems.

## Summary

The first learning activity of this unit introduced the concept of integration as a converse operation to differentiation. We therefore obtained an anti-derivative as a function whose derivative is the given function, sometimes also referred to as the integrand. In this learning activity the concept of a definite integral is introduced. Here, the integral produces, not a function, but a real number.

Analytical and numerical methods of evaluating definite integrals are presented.

Improper integrals are defined and indication is given how to determine their convergence. Applications of definite and improper integrals are also given.

## Compulsory Reading

All of the readings for the module come from Open Source text books. This means that the authors have made them available for any to use them without charge.

We have provided complete copies of these texts on the CD accompanying this course.

**The Calculus Bible, Prof. G.S. Gill:** Brigham Young University, Maths Department.

Brigham Young University – USA.

The definite integral: Chapter 5 Section 5.2

Techniques for integration: Chapter 6

Improper integrals: Chapter 72.3b



## Internet and Software Resources

### Web References

#### Asguru (Visited 07.11.06)

<http://www.bbc.co.uk/education/asguru/maths/12methods/04integration/21definite/index.shtml>

This link gives access to a two page article on the definite integral.

#### Nrich (Visited 07.11.06)

<http://www.nrich.maths.org/public/leg.php?code=-211>

Try these puzzles on the theme of integration.

#### Wolfram Mathworld (Visited 07.11.06)

<http://mathworld.wolfram.com/>

Search for: definite integral [If you are using this document on a computer, then the links can be clicked directly].

#### Wikipedia (visited 07.11.06)

[http://en.wikipedia.org/wiki/Main\\_Page](http://en.wikipedia.org/wiki/Main_Page)

### Key Concepts

#### Definite integral

The definite integral of a **continuous** function  $f(x)$ , defined over a **bounded**

n

interval  $[a, b]$ , is the **limit of a sequence of Riemann sums**  $S =$

x

$\Delta f(\xi)$

n

$\sum_{i=1}^n \Delta f(\xi_i)$

$i = 1$

obtained by subdividing (partitioning) the interval  $[ a, b ]$  into a number  $n$  of

subintervals in such a way that as the largest subinterval  $\Delta x = x_i - x_{i-1}$

$i$

$i$

$i - 1$  in

the sequence of partitions also tends to zero. The definite integral of  $f ( x )$  over

$b$

$[ a, b ]$  is denoted by

$\int_a^b f(x)dx$

$\int$

.

$a$



African Virtual University 59

### **Improper Integral**

An improper integral is a definite integral in which either the interval  $[ a, b ]$  is

not bounded  $a = -\infty$  ,  $b = \infty$ , the function (integrand)  $f ( x )$  is discontinuous

at some points in the interval, or both.

### **Key Theorems and/or Principles**

#### **Integrability of a function**

Every function  $f$  that is continuous on a closed and bounded interval  $[ a, b ]$  is

integrable on  $[ a, b ]$  .

# Additivity of integrals

If  $f$  is integrable on  $[a, b]$  and  $a < c < b$  then

$b$

$c$

$b$

$\int_a^b f(x) dx =$

$\int_a^c f(x) dx +$

$\int_c^b f(x) dx$

$\int$

$\int$

$\int$

$a$

$a$

$c$

## order property

$b$

$b$

If  $f(x) \leq g(x)$  for all  $x \in [a, ]$

$b$ , then

$\int_a^b f(x) dx \leq$

$\int_a^b g(x) dx$

$\int$

$\int$

$a$

$a$

## Mean Value Theorem for Integrals

If  $f$  is integrable on  $[a, b]$  then there exists a number  $c \in [a, ]$

$b$  such that

$b$

$1$

$f$

$b$

$$\int_a^b f(x) dx = (b - a) f(c)$$

$\int$

. The value  $f(c) =$

$$\frac{1}{b - a} \int_a^b f(x) dx$$

$\int$

is called the

$a$

$a$

$b - a$

average (mean) value of the integral.

## Fundamental Theorem of Calculus

$x$

If  $f$  is continuous on  $[a, b]$  and  $F(x) =$

$$\int_a^x f(t) dt$$

$\int$

for each  $x$  in  $[a, b]$ , then

$a$

$F(x)$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  and  $dF = f(x)$ . In other words,  $F(x)$  is an anti-derivative of  $f(x)$ .



$dx$



African Virtual University 60

The proof of this theorem is relatively straightforward. It is based on the Mean Value Theorem for integrals stated above and on the definition of a continuous function  $f(x)$  on a bounded interval.



African Virtual University 61

### Learning Activity

Having defined the concept of a definite integral we now need to learn how to evaluate a definite integral.

Fortunately the concept of a primitive function introduced earlier gives us the answer. We state and show this important result in the form of a theorem.

**Theorem:** If  $f(x)$  is continuous on a closed and bounded interval  $[a, b]$ , and

,

$b$

$g(x) = f(x)$ , then

$\int_a^b f(x)dx = g(b) - g(a)$

$b - g(a)$

$\int$

Using the Fundamental theorem of Calculus we define a function

$x$

$$G(x) =$$

$$\int_a^x f(t) dt$$

$\int$

$a$

Note that

$$G'(x) = f(x) \text{ and } G(a) = 0$$

$G(x) = \int_a^x f(t) dt$ . Therefore,  $G$  and  $g$  are both anti-

derivatives (primitive functions) of  $f$ . Therefore they differ only by a constant

and hence  $G(x) = g(x) + C \forall x$ . But  $G(a) = g(a) + C = 0$ . This gives the value  $C = -g(a)$ .

We can now write  $G(x)$  as  $G(x) = \int_a^x f(t) dt - g(a)$ . Evaluation of  $G(x)$  at  $x = b$

leads to the desired result:  $G(b) = \int_a^b f(t) dt - g(a)$

$$= \int_a^b f(t) dt - g(a)$$

$$= \int_a^b f(t) dt - g(a)$$

$$= \int_a^b f(t) dt - g(a)$$

$\int$

$a$

$b$



**Mathematical Problem**

**Evaluation of Definite integrals**

In the first Learning Activity we learnt different methods of finding primitive functions (anti-derivatives) of given functions. Using the result from the above theorem we are now able to evaluate any definite integral provided we can find a primitive function of the integrand.

Definite integrals can be evaluated in two ways. One method is using the anti-derivative, where it is known. A second method is by applying numerical methods.

**Evaluation of Definite integrals Using Anti-derivatives**

b

If the anti-derivative of  $f(x)$  is  $F(x)$ , then  $\int_a^b f(x)dx = F(b) - F(a)$ .

a

3

x

For example, since the anti-derivative of

2

$f(x) = x$  is  $F(x) =$

, it follows

3

3

2

19

that

$$\int_2^3 x \, dx = F(3) - F(2) =$$

$\int$

F

.

2

3

**Do THIS**

$\pi$

Evaluate the definite integral  $\int_0^{\pi} 2 \cos(x) \, dx$ .

0

### **Evaluation of Definite integrals Using Numerical Methods**

Numerical methods are applied in evaluating definite integrals under two circumstances:

(i) The anti-derivative of the integrand  $f(x)$  may not exist, although the  $f(x)$  may perfectly well be integrable. A typical example of this is the function

2

$f(x)$

$x$

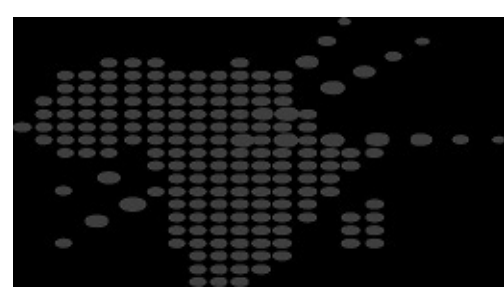
$= e^{-x^2}$ . This exponential function with an exponent 2

$x$  is integrable over

any closed and bounded interval  $[a, b]$ . However, there is no function  $F(x)$

whose derivative gives  $f(x)$ .

(ii) Some functions  $f(x)$  may have an anti-derivative  $F(x)$  which is very difficult to evaluate at a given point. For, example, the anti-derivative of the



African Virtual University 63

1

function  $f(x) =$

is  $F(x) = \tan^{-1}(x)$  and the evaluation of  $\tan^{-1}(x)$

2

1

+  $x$

is not a straightforward matter.

In both of the above two cases, the value of the integral can only be approximated using an appropriate numerical method.

There are many numerical methods for approximating definite integrals. Two of these methods are discussed in your Self Instructional Material (S.G. Gills), namely the Trapezium (Trapezoidal) rule and the Simpson Rule.

**Do THIS**

Study carefully the derivation of the two formulas

$n-1$

$x_n$

$h$

(i) Trapezium Rule

$\int f(x)dx = ($

$\{ f$   
 $f$   
 $2$   
 $f$   
 $0 +$   
 $n)$   
 $\int$   
 $+$   
 $x$   
 $\sum k \}$   
 $0$   
 $2$   
 $k=1$   
 $n$   
 $n-$   
 $x$   
 $1$   
 $n$   
 $h$

(ii) Simpson's Rule

$f(x)dx = ($

$\{ f$   
 $4$   
 $2$   
 $0 + f$

f2 1

f

n )

f

+

x

$\sum k +$

-

$\sum 2k \}$

0

3

k=1

k=1

1

And apply them on the integral  $\int \sin(x) dx$  using  $n = 10$  ( $h = 0.1$ ).

0

### Evaluation of improper integrals

b

A definite integral  $\int_a^b f(x) dx$  may be improper due to one or both endpoints

a

of the range of integration being infinite, or due to the integrand  $f(x)$  being

undefined at some point  $c \in [a, ]$

$b$ , or indeed when both situations prevail.

Whichever is the case, **evaluation of an improper integral always involves**

**taking a limit.** If the limit exists, then the improper integral is said to converge,

and the limit is then the value of the integral. Otherwise, the improper integral is said to be divergent.

**Example:**

$\infty$   
The improper integral

$$e^{-x} dx$$

$\int$   
can be evaluated by finding the following

0

b

limit:  $\lim$

$$e^{-x} dx .$$

$$\rightarrow \infty \int$$

-

b

0



African Virtual University 64

It is extremely important to observe this rule of taking limits. Some learners, knowing anti-derivative of

$$-x$$

$f(x) = e^{-x}$  to be the function

$$-x$$



$F(x) = -e^{-x}$  are tempted to

$\infty$

evaluate the integral in the same way they evaluate definite integrals by writing  $e^{-x} dx$

$\int$

$0$

$= F(\infty) - F(0) = [-e^{-x}]_0^{\infty}$

$0$

)

$0$

$-e^{-\infty} - (-e^{-0}) = 0 - (-1) = 1$ . This is wrong. Infinity ( $\infty$ ) is not a number, hence

it cannot be substituted into a function.

**Do THIS**

$\infty$

$dx$

Evaluate  $\int_0^{\infty} e^{-x} dx$

if it exists.

$0 \int_0^{\infty} (1+x)^{-2} dx$

## Applications of Definite and improper integrals

We mention, in passing, that definite and improper integrals are very useful in evaluating some common mathematical problems. We shall mention two common cases.

### (i) Calculating areas

**The definite integral** is often used in calculating areas under a curve or enclosed between two given curves. Indeed, many texts on this subject use this to introduce the subject of finding areas. The learner is strongly encouraged to look this up in

the Main Reference (S.G. Gills) and solve some relevant problems there in.

**(ii) Testing convergence of infinite series**

**The improper integral** is often used in deciding whether or not some infinite series converge. This be dealt with in more detail in the next Unit (Unit 3) of this Module.

**Exercise (From Gill: The Calculus Bible)**

Solve as many problems as you can in the following Exercises:

Exercise 5.2

pp 209-210

Exercise 5.3

pp 214-215

Exercise 5.4

p 229

Exercise 5.8

pp 264-266

Exercise 7.2

pp 299-304



African Virtual University 65

**Mathematics Module 3: Calculus**

**Unit 3: Sequences and Series (20 Hrs)**

**Learning Activity No.1**

**TITLE: Sequences and Series (20 Hrs)**

**Note:** Due to the relatively short time allocated to this unit, and also because of the close link that there is between sequences and infinite series, we shall present the material for this Unit using a single Learning Activity.

### **Specific Learning Objectives**

At the end of this activity the learner will be able to:

(iii)

Explain what is meant by a sequence.

(iv)

Determine whether or not a given sequence converges.

(iii)

Evaluate limits of convergent sequences.

(iv)

Explain what is meant by an infinite series.

(v)

Explain what is meant by a power series.

(vi)

Find the  $n$ th partial sum of a given series.

(vii)

Determine whether or not a given series converges.

(viii)

Find sums of convergent infinite series.

(ix)

Apply various convergence tests.

(x)

Use the knowledge of sequences and series to solve some relevant

problems in Mathematics and in Science.

## Summary

Sequences and infinite series play an important role in analyzing various physical phenomena. Iteration methods, which are indispensable in solving a large number of mathematical problems, are a major source of sequences. The need to find out the behavior of a function in a close neighborhood of a given point in its domain leads to power series expansion of such functions.

In this learning activity we shall define sequences and infinite series, establish a relationship between the two concepts, determine criteria for their convergence and learn how to find their respective limits, where they exist. The special case of power series, including the associated concepts of interval and radius of convergence will be covered. Applications of series will also be touched upon.



African Virtual University 66

## Compulsory Reading

All of the readings for the module come from Open Source text books. This means that the authors have made them available for any to use them without charge.

We have provided complete copies of these texts on the CD accompanying this course.

**The Calculus Bible, Prof. G.S. Gill:** Brigham Young University, Maths Department.

Brigham Young University – USA.

# Chapter 8

## internet and Software Resources

### Software

Use **wxMaxima** to check your solution to integration problems.

You should use the **integrate** function. It works like this:

- $\text{Integrate}(x^2,x)$  will integrate the function  $x^2$  with respect to  $x$ .
- $\text{Integrate}(x^2,x,0,2)$  will complete a definite integration of  $x^2$  with respect to  $x$  with limits from  $x = 0$  to  $x = 2$ .

Look for more advanced functions in the **wxMaxima** manual.

### **Asguru (Visited 07.11.06)**

<http://www.bbc.co.uk/education/asguru/maths/13pure/03sequences/index.shtml>

This link gives access to a number of pages which introduce sequences and series.

Click the links in turn to work through the different sections. When you have read to the bottom of a page click the large right arrow. This will take you to the next page in the section.

Many of the sections contain interactive activities. Read the instructions carefully and explore the ideas.

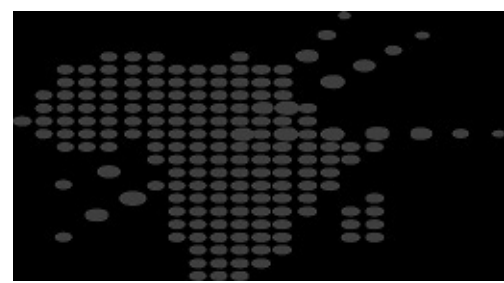
Some sections contain interactive tests and exercises. Use these to check your understanding.

### **Nrich (Visited 07.11.06)**

<http://www.nrich.maths.org/public/index.php>

Type sequences and series into the search box in the top right hand corner of the page.

Try these sequences and series puzzles.



African Virtual University 67

There are a very large number to choose from. We recommend you only look at stage 4 and 5 puzzles. Try the first one(s) that look interesting.

### **Wolfram Mathworld (Visited 07.11.06)**

<http://mathworld.wolfram.com/Series.html>

Read this entry for Series.

Follow links to explain specific concepts as you need to.

### **Wikipedia (visited 07.11.06)**

[http://en.wikipedia.org/wiki/Sequence\\_and\\_series](http://en.wikipedia.org/wiki/Sequence_and_series)

Read this entry for Sequences and Series.

Follow links to explain specific concepts as you need to.

### **Key Concepts**

#### **Sequence**

A sequence is an unending list of objects (real numbers)  $a_1, a_2, a_3, a_4, \dots$ . The 1

2

3

4

number  $a_t$  is called the  $t$

$n$ th term of the sequence, and the sequence is denoted

$\{a_n\}$

by  $\{a_n\}$ .

$\{a_n\}$

## Convergence of a Sequence

A sequence  $\{a_n\}$  is said to converge if  $\lim_{n \rightarrow \infty} a_n$  exists. If the limit does not exist

$n\}$

$n$

$n \rightarrow \infty$

then it is said to diverge (or is divergent).

## Infinite Series

$\infty$

An infinite series, written

$a$

$\sum_{k=1}^{\infty} a_k$ , is a sum of elements of a sequence  $\{a_n\}$ .

$n\}$

$k$

$k=1$



African Virtual University 68

## Partial Sums

The  $n$ th

partial sum of a series, written  $S_n$  is the sum of the first  $n$  terms:

$n$

$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$

$n$

1

2

3

$n$

### Arithmetic series

An arithmetic series is a series in which the difference between any two consecutive terms is a constant number  $d$ . If the first term of the series is  $a$  then the

$t$

$n$ th partial sum of such a series is given by

$n$

$S =$

$a + n - d$

$n$

(2

(

1) )

2

### Geometric series

A geometric series is a series in which, except for the first term, each term is a constant multiple of the preceding term. If the first term is  $a$  and the common factor is  $r$  then the  $t$

$n$ th partial sum of the geometric series is given by

$n+1$

$[1 - r$

$S = a$

$]$ .



n

| r 1 |

-

[

]

### Power series

A power series in  $x$  is an infinite series whose general term involves a power of

$\infty$

the continuous variable  $x$ . For example, a

$a_0 + a_1 x +$

$a_2 x^2 +$

$\dots$

$+ a_k x^k +$

$a_{k+1} x^{k+1} +$

$a_{k+2} x^{k+2} +$

$a_{k+3} x^{k+3} +$

$a_{k+4} x^{k+4} +$

$a_{k+5} x^{k+5} +$

$a_{k+6} x^{k+6} +$

$a_{k+7} x^{k+7} +$

$a_{k+8} x^{k+8} +$

$a_{k+9} x^{k+9} +$

$a_{k+10} x^{k+10} + \dots = \sum_{k=0}^{\infty} a_k x^k$

is a power series.

## Interval of convergence of a power series

The interval of convergence of a power series is an interval (finite or infinite)

–  $R < x < R$  in which the series converges.

## Radius of convergence of a power series

The radius of convergence of a power series is the number  $R$  appearing in the interval of convergence.



African Virtual University 69

## Key Theorems and/or Principles

### (i) Sequences

• A monotone sequence  $\{ a_n \}$  converges if it is bounded.

$n \}$

• A sequence  $\{ a_n \}$  converges only if it is a Cauchy sequence

$n \}$

### (ii) Cauchy Sequence

A sequence  $\{ a_n \}$  is called a **Cauchy Sequence** if and only if for every positive

$n \}$

number  $\epsilon > 0$ , however small, there exists a corresponding positive integer

$N(\epsilon)$ , such that a

$a_n$

for all  $n, m > N(\epsilon)$ ,

$|a_n - a_m| < \epsilon$ . It can be proved that if a

$n, m > N(\epsilon)$

$$n < \epsilon$$

sequence is a Cauchy sequence, then it converges.

### (iii) Series

$\infty$

• **Divergence Test:** If the series  $\sum a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

. If

$$\lim_{n \rightarrow \infty} a_n \neq 0$$

$n$

$\rightarrow$

$n \rightarrow \infty$

$$k = 1$$

$\lim_{n \rightarrow \infty} a_n$

then the series diverges. We note, in passing, that  $\lim_{n \rightarrow \infty} a_n = 0$

is only

$$n = 0$$

$$n \neq 0$$

$\rightarrow$

$n \rightarrow \infty$

$\rightarrow$

$n \rightarrow \infty$

a necessary condition for a series to converge but it is not a sufficient condition.

This means that there are divergent series for which the condition applies. Indeed,

$\infty$

1

one simple series, the harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$  can be shown to be divergent al-

k

k = 1

( 1

though clearly lim

)

| | = 0.

n → ∞ ( n )

• **Integral Test:** Let  $f(x)$  be continuous and positive for all  $x \in$ ,

1

$[\infty)$ . Then

$\infty$

the infinite series  $\sum f(n)$  converges if, and only if, the improper integral

k

converges.

= 1

• **Comparison Test:** Let  $0 < a < b$  for all  $n > 1$ . Then,

n

n

$\infty$

$\infty$

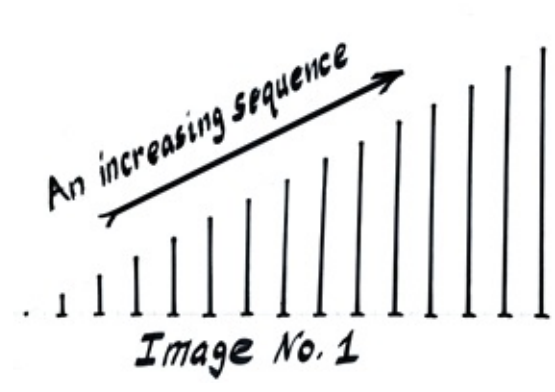
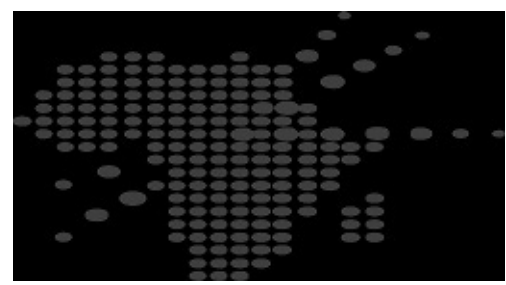
if  $\sum b$  converges then a also converges, and

n

$\sum n$

k = 1

k = 1



African Virtual University 70

$\infty$

$\infty$

if  $\sum a$  diverges then  $b$  also converges.

$n$

$\sum n$

$k=1$

$k=1$

$a$

• **Ratio Test:** If  $a$

for all  $n \geq 1$  and if

$n+1$

$\lim$

$= L$ , then the series

$n > 0$

$n$

$\rightarrow \infty$  an

converges if  $L < 1$  and diverges if  $L > 1$ . If  $L = 1$  the test fails.

( 1 )

• **n - th Root Test:** If  $a$

for all  $n$  and if  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$  then

$n > 0$

$n$

$n \rightarrow \infty$

(

)

$\infty$

the series  $\sum a_k$  converges if  $L < 1$  and diverges if  $L > 1$ . If  $L = 1$  then  $n$

$k$

$= 1$

the test fails.

**Learning Activity**

# introduction

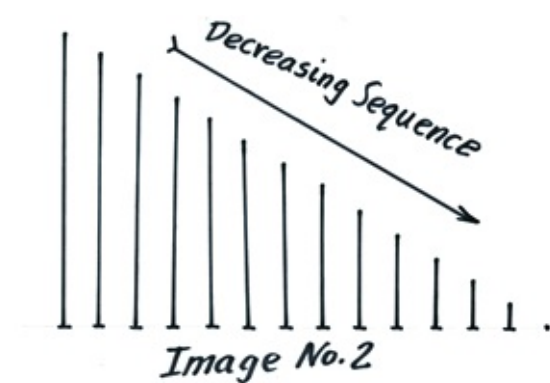
1. Look carefully at the images (  $I$  and  $I$  ) of objects (sticks) arranged sequen-

1

2

tially.

Both are meant to represent the first few terms of an infinite sequence of numbers



African Virtual University 71

$\{ a_n \}$ . In each case the arrow points in the direction of increasing values of the

$n$  }

index  $n$  of the elements of the sequence.

## Questions

(i) Is the sequence  $\{ a_n \}$  represented by the Image  $I$  increasing or de-

$n$  }

1

creasing?

(ii) Is the sequence converging or not converging (diverging)?

Pose and attempt to answer the same two questions for the sequence represented

by the image  $I$ .

2

2. Consider the sequence of numbers whose first five terms are shown here:

1

1

1

1

1, , , , ...

2

3

4

5



African Virtual University 72

### Key Definitions

A clear understanding by the learner, of terms used in this learning activity, is very essential. The learner is therefore strongly advised to learn and carefully note the following definitions.

#### (i) Sequences

• A sequence  $\{ a_n \}$  is said to be **monotone** if it is increasing, decreasing, non-

increasing or non-decreasing.

increasing or non-decreasing.



• A sequence  $\{ a_n \}$  is said to be **bounded** if there exist two numbers  $m, M$

such that  $m \leq a_n$

$\leq M$  for all  $n$  larger than some integer  $N$ .

$n \leq$

## (ii) Series

$\infty$

• **Absolute convergence:** A series  $\sum_{k=1}^{\infty} a_k$  is said to converge absolutely if the

$k$

$k=1$

$\infty$

series  $\sum_{k=1}^{\infty} |a_k|$  converges.

$k$

$k=1$

$\infty$

• **Conditional convergence:** If the series  $\sum_{k=1}^{\infty} a_k$  converges but the series

$k$

$k=1$

$\infty$

$\infty$

$\sum_{k=1}^{\infty} |a_k|$  diverges, then  $\sum_{k=1}^{\infty} a_k$  is said to converge conditionally.

$k$

$\sum_{k=1}^{\infty} k$

$k=1$

$k=1$

$\infty$

• **Positive term series:** If  $a$

for all  $n$ , then the series

$a$  is called

$n > 0$

$\sum k$

$k$

a positive term series.

$=1$

• **Alternating series:** If  $a$

for all  $n$  then the series

$n > 0$

$\infty$

$\sum (-)^{k+1}$

1

$a$

$a$

$a$

$a$

$a$

$k =$

.....

1 -

2 +

3 -

4 +

k=1

or

$\infty$

$\sum(-)$

1 k a

a

a

a

a

k = -

.....

1 +

2 -

3 +

4 -

k=1

is called an alternating series.



African Virtual University 73

### Mathematical Analysis

The principal reference text for the learner in the topic of Sequences and Series

is **S.G. Gill**. The material is contained in Chapter 8 (pp 315 – 354)

## Sequences

For a sequence, the mathematical problem is

**Either:** to determine whether it converges or diverges,

**or:** to find the limit if it converges.

After giving a mathematical definition of the concept of convergence of a sequence, a set of standard algebraic properties of convergent sequences is given on page 316. The set includes limits of sums, differences, products and quotients of convergent sequences. The learner is urged to know those properties and read carefully the proofs that immediately follow (pp 316 – 319).

Conditions for convergence of a sequence are very well spelt out in the main reference sighted above.

### Do THIS

Study carefully the set of key definitions given on page 320 of the text. The definitions are immediately followed by a key theorem which gives criteria for convergence. In a group work, go over the given proofs to ensure you understand and accept the premises (conditions stated) and the conclusion of the theorem.

**Error Alert:** Please note an unfortunate printing error that appears on page 323

n  
of the reference, where the t

n h partial sum  $\sum t$  is referred to as an infinite

k

k =1

$\infty$

series. This should be corrected to  $\sum t$  .

k

$$k = 1$$

## Do THIS

On the basis of the definition of convergence given on page 315, the properties of convergent sequences given on page 316 and the definitions of terms and convergence criteria stated and proved starting from page 320, **solve as many problems as you can** from Exercise 8.1 (pp 326 – 327).



African Virtual University 74

## infinite series

$\infty$

In general, infinite series are derived from sequences. An infinite series  $\sum a_k$  can

$k$

$k = 1$

be viewed as a sum of the terms of a sequence  $\{a_k\}$  and convergence of the series

$k \}$

is defined in terms of convergence of the associated sequence  $\{S_1, S_2, S_3, S_4,$

1

2

3

4

}

,...

n

of the series, where  $S$

$a_n$  is the sum of the first  $n$  terms of the series, also

$$a_n = \sum_{k=1}^n k$$

$k$

$=1$

called the  $n$ -th partial sum.

### Example

{

1

}

Let  $\{$

$\}$  be a sequence of numbers from which we can form the se-

$\{$  (

$n(n+1)$

$\}$

$\infty$

1

series  $\sum$

. To determine whether or not this series converges, we need

$k$   $k$

$k$

(

)

1

1

+  
=  
n  
1  
1  
1  
1

to form its  $n - th$  partial sum  $S$

. Since

=  
-

$n = \sum ($

)

1

$k(k + )$

1

$k k + 1$

1  $k k$

$k$

+

=

n

1

( 1

1 )

1

one finds  $S$

, and since

$n =$

$=$

$\sum$

$\sum | -$

$| = 1 -$

$k \geq 1$   $k(k$

$k k$

$n$

$=$

$+) )$

1

$\setminus$

$+1)$

$+1$

$($

1

now  $\lim S$

, we conclude that the infinite series

$n = \lim 1$

$\setminus$

$| -$

$| = 1$



→

$n \rightarrow \infty$

→

$n \rightarrow \infty$

$(n+1)$

$\infty$

1

$\sum$

converges, and that its sum is unity.

$k$

$k$

(

)

1

1

+

=

Presentation of the material on infinite series is done in **four major stages**.

• **Section 8.4** (pp 329 – 334) is devoted to **positive term series**

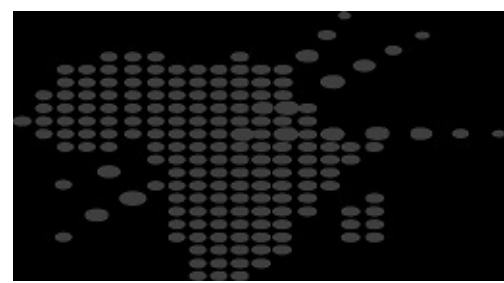
Under positive term series some very important theorems are stated, including:

(i) The comparison test (page 339)

(ii) The ratio test (page 331) and

(iii) The  $n$  –  $th$  root test.

These are stated and quite elegantly proved.



African Virtual University 75

## Do THIS

Study carefully the assumptions made for each of the tests, read carefully the proofs given and then check the level of your understanding by solving as problems as you can in **Exercise 8.2 (pp 335 – 341)**.

- **Section 8.5** (pp 341 - 346) presents **alternating series**

Alternating series are quite common when evaluating some power series at a fixed point. Important results related to alternating series include the twin concepts of **absolute** and **conditional** convergence associated with such series.

A classical example of an alternating series which is conditionally convergent

$k + 1$

$\infty$

1

is the series  $\sum (-)$

1

which can be shown to converge. However, the series

$k$

$=1$

$k$

obtained by taking absolute values of the terms is the harmonic series which we know diverges. Its being divergent can easily be proved using the integral test.

- **Section 8.6** (pp 347 – 354) deals with **power series**

A power series is a series of the form

$$\sum_{k=0}^{\infty} a_k x^k$$
$$\sum_{k=0}^{\infty} a_k (x - c)^k$$
$$\sum_{k=0}^{\infty} a_k x^k$$
$$\sum_{k=0}^{\infty} a_k (x - c)^k$$
$$\sum_{k=0}^{\infty} a_k x^k$$

**or**

$$\sum_{k=0}^{\infty} a_k (x - c)^k$$

$a_k (x - c)^k$ , where  $c$  is a constant.

$$\sum_{k=0}^{\infty} a_k (x - c)^k$$
$$\sum_{k=1}^{\infty} a_k x^k$$
$$\sum_{k=1}^{\infty} a_k (x - c)^k$$
$$\sum_{k=1}^{\infty} a_k x^k$$

The **convergence of a power series** can be established by applying the **ratio**

**test**. Here one considers

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right|$$
$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} x \right|$$
$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| |x|$$
$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| |x|$$
$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| |x|$$
$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| |x|$$
$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| |x|$$
$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| |x|$$
$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| |x|$$
$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| |x|$$

$\lim$

+

or

n

$x - c$

1

$\lim$

+

•

n

$n \rightarrow \infty$

a x

$n \rightarrow \infty a$

$n \rightarrow \infty a$

n

n

n

a

If

$n+1$

$\lim$

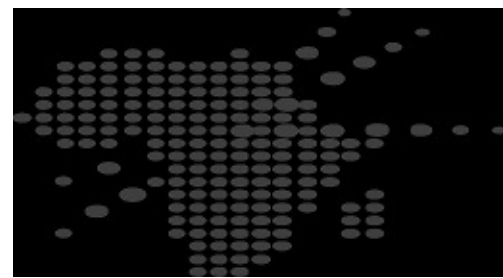
= L then for convergence one demands that  $x L < 1$  or

$n \rightarrow \infty a$

n

$x - c L < 1$ . In turn, these conditions lead to the following restrictions on the

possible values of  $x$  for which the power series will converge:



African Virtual University 76

1

1

$x < c$  or  $x - c <$

.

$L$

$L$

Each of these inequalities defines an interval within which  $x$  must lie for the power series to converge. The intervals are:

1

1

1

1

-

$< x <$

or  $c -$

$< x < c +$

$L$

$L$

$L$

$L$

These are known as **intervals of convergence** of the respective power series and

1

1

the quantities  $R =$

or  $R = c +$

(  $L \neq 0$  ) are called **radius of conver-**

$L$

$L$

**gence.**

**Question:** If  $L = 0$  , for what values of  $x$  do the series converge? And what

happens if  $L = \infty$  ?

**Section 8.7** (pp 354 – 360) presents **Taylor series expansion**

These are presented as special cases of power series, and all the results established

for general power series apply.

**Software Activity**

Check your Taylor series with mxMaxima.

For example:

Type `taylor(x^3,x,1,4)` and press RETURN

This will show the first 4 terms of a Taylor expansion of  $(x - 1)^3$

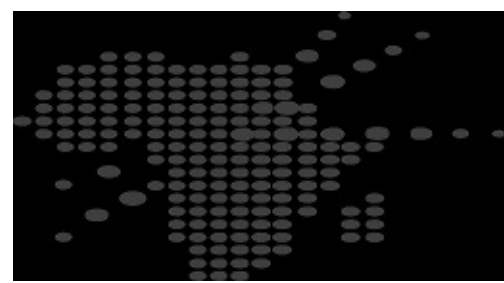
It will give:

$1+3*(x-1)+3*(x-1)^2+(x-1)^3+...$

**Exercises (From Gill: The Calculus Bible)**

Exercise 8.1

page 326



African Virtual University 77

**XV.**

### **Synthesis of the Module**

Completion of Unit Four marks the end of this module. At this juncture, and before presenting oneself for the summative evaluation, it is desirable and appropriate that the learner reflects back on the mission, objectives, activities and attempts to form a global picture of what he/she is expected to have achieved as a result of the collective time and efforts invested in the learning process.

As laid out in the module overview in Section 6, the learner is expected to have acquired knowledge of the basic concepts of differential and integral calculus of functions of one and several independent variables. Inherently imbedded in the key word “**calculus**” is the underlying concept of “**limits**” which underpins the entire spectrum of coverage of the module.

The learner is now expected to be able to comfortably define the limit concept, evaluate limits of functions, sequences and infinite series. One is also expected to appreciate the use of the limit concept in defining the concepts of continuity, differentiation (ordinary and partial) and integration (Riemann integral). Thus, the signposts (salient features) on the roadmap for this module are **limits, continuity, differentiation and integration** as applied to functions of both single and several (specifically two) independent variables.

The key areas of application of the knowledge acquired are in determining local maxima and minima, areas, volumes, lengths of curves, rates of change, moments

of inertia and centers of mass.

The module has been structured with a view to escorting and guiding the learner through the material with carefully selected examples and references to the core references. The degree of mastery of the module contents will largely depend on the learner's deliberate and planned efforts to monitor his/her progress by solving the numerous self-exercise problems that are pointed out or given throughout the module.



African Virtual University 78

## **Mathematics Module 3: Calculus**

### **Unit 4: Calculus of Functions of Several Variables (30 Hrs)**

#### **Activity 1: Limits, Continuity and Partial Derivatives (11 Hrs)**

##### **Specific Learning Objectives**

At the end of this activity the learner should be able to:

- Find out whether or not a limit exists;
- Evaluate limits of functions of several variables;
- Determine continuity of a function at a point and over an interval.
- Find partial derivatives of any order.
- Expand a function in a Taylor series about a given point

##### **Summary**

The first three units of this module dealt exclusively with the elementary differential and integral calculus of functions of a single independent variable, symbolically expressed by the equation  $y = f(x)$ . However, only a very small



minority of real life problems may be adequately modeled mathematically using such functions. Mathematical formulations of many physical problems involve two, three or even more variables.

In this unit, and without loss of generality, we shall present the elementary differential and integral calculus of functions of two independent variables. Specifically, in this learning activity, we shall discuss the concepts of limit, continuity and partial differentiation of functions of two independent variables.

### **Compulsory Reading**

All of the readings for the module come from Open Source text books. This means that the authors have made them available for any to use them without charge.

We have provided complete copies of these texts on the CD accompanying this course.

**The Calculus Bible, Prof. G.S. Gill:** Brigham Young University, Maths Department, Brigham Young University – USA.



African Virtual University 79

### **Internet and Software Resources**

#### **Software**

Extend your expertise with mxMaxima. Investigate any functions that you find.

Check in the wxmaxima manual to find if wxMaxima can do them. If it can practice and explore.

#### **Wolfram Mathworld (Visited 07.11.06)**

<http://mathworld.wolfram.com/PartialDerivative.html>

Read this entry for Partial Derivatives.

Follow links to explain specific concepts as you need to.

### **Wikipedia (visited 07.11.06)**

[http://en.wikipedia.org/wiki/Partial\\_derivative](http://en.wikipedia.org/wiki/Partial_derivative)

Read this entry for Partial Derivatives.

Follow links to explain specific concepts as you need to.

Use Wikipedia and MathWorld to look up any key technical terms that you come across in this unit.

## **Key Concepts**

### **Domain and Range**

If  $z = f(x, y)$  is a function of two independent variables then the set  $D = \{$

$(x,$

$y) \in \mathbb{R}^2\}$  of points for which the function  $z = f(x, y)$  is defined is

called the **Domain of  $z$** , and the set  $R = \{z = f(x, y) : (x, y) \in D\}$  of values

of the function at points in the domain is called the **Range of  $z$** .

### **Graph of a function**

The graph of a function  $z = f(x, y)$  is the set  $G = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D\}$

While the graph of a function of a single variable is a curve in the  $xy$ -plane, the

graph of a function of two independent variables is usually a surface.



A level curve of a function of two variables  $z = f(x, y)$  is a curve with an equation of the form  $f(x, y) = C$ , where  $C$  is some fixed value of  $z$ .

### Limit at a point

Let  $(a, b)$  be a fixed point in the domain of the function  $z = f(x, y)$  and  $L$  be a real number.  $L$  is said to be the limit of  $f(x, y)$  as  $(x, y)$  approaches  $(a, b)$  written

$\lim$

$f(x, y) = L$ , if for every small number  $\epsilon > 0$  one can find

$(x, y) \rightarrow (a, b)$

a corresponding number  $\delta = \delta(\epsilon)$

$\epsilon$  such that  $f(x, y) - L < \epsilon$  whenever

$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$

)

$b < \delta$

One also writes

$\lim_{(x, y) \rightarrow (a, b)} f(x, y) = L$  or  $f(x, y)$  approaches  $L$  as  $(x, y)$  approaches  $(a, b)$ .

$x \rightarrow a$

$y \rightarrow b$

### Continuity

A function  $z = f(x, y)$  is continuous at a point  $(a, b) \in D$  if three conditions are satisfied:

(i)  $f(x, y)$  is defined at  $(a, b)$  [the value  $f(a, b)$  exists], (ii)  $f(x, y)$  has a limit  $L$  as  $(x, y)$  approaches  $(a, b)$ , and (iii) The limit  $L$  and the value  $f(a, b)$  of the function are equal.

This definition can be summed up by writing .

$\lim$

$= f(a, b)$

$b$

$(x, y) \rightarrow (a, b)$

## Partial derivative

A partial derivative of a function of several variables is the derivative of the given function with respect to one of the several independent variables, treating all the other independent variables as if they were real constants. For, example, if

$f(x, y, z) = x^2 y + 3xz^2 - xyz$  is a function of the three independent variables,

$x$ ,  $y$ , and  $z$ , then the partial derivative of  $f$  with respect to the variable  $x$  is the

function  $g(x, y, z) = 2xy + z^2$

3

$- yz$ .



African Virtual University 81

## Key Theorems and/or Principles

### Criteria for Nonexistence of a Limit

If a function  $f(x, y)$  has different limits at a point  $(a, b)$  as the point  $(x, y)$  approaches point  $(a, b)$  along different paths (directions), then the function has

no limit at point  $(a, b)$ .

**Remark:** Taking limits along some specific paths is not a viable strategy for

showing that a limit exists. However, according to this theorem, such an approach

may be used to show that a limit does not exist.

### Sufficient Condition for Existence of a Limit

Let  $D$  be the domain of a function  $f(x, y)$  and assume that there exists a number  $L$  such that  $f(x, y) - L \leq g(x, y) \forall (x, y)$  that satisfy the inequality  $0 < (x -$

2

$$a) + (y - 2)$$

)

$$b < \delta .$$

If

$$\lim g( ,$$

$$x y) = 0, \text{ then } \lim$$

$$f(x, y) = L .$$

$$(x, y) \rightarrow (a, b)$$

$$(x, y) \rightarrow (a, b)$$

### **Continuity of Composite Functions**

If  $f(x, y)$  is continuous at  $(a, b)$  and  $g(t)$  is continuous at  $f(a, b)$ , then the composite function  $h(x, y) = g(f(x, y))$  is continuous at  $(a, b)$ .

### **Chain Rule for Partial Differentiation**

**(a)** If  $w = f(x, y)$  is a differentiable function of  $(x, y)$  and  $x = x(t)$ ,  $y = y(t)$  are differentiable functions of  $t$ , then

$$dw$$

$$w$$

$$\partial dx$$

$$w$$

$$\partial dy$$

$$=$$

$$+$$

$$dt$$

$$x$$

$$\partial dt$$

$$y$$

$\partial dt$

**(b)** If  $w = f(x, y)$  is a differentiable function of  $(x, y)$  and  $x = x(u, v)$ ,  $y = y(u, v)$  are differentiable functions of  $(u, v)$ , then  $w$

$\partial$

$w$

$\partial$

$x$

$\partial$

$w$

$\partial$

$y$

$\partial$

$\partial w$

$\partial w \partial x$

$\partial w \partial y$

$=$

$+$

and

$=$

$+$

$\cdot$

$u$

$\partial$

$x$

$\partial$

$u$

$\partial$

$y$

$\partial$

$u$

$\partial$

$\partial v$

$\partial_x \partial v$

$\partial_y \partial v$



African Virtual University 82

### Equality of Mixed Partial Derivatives

$\partial^2 f$

$\partial^2 f$

If the mixed second partial derivatives

$= f$  and

$= f$  are

$xy$

$\partial_x \partial_y$

$yx$

$\partial_y \partial_x$

continuous on an open disc containing point  $(a, b)$ , then  $f(a, )$

$b$

$f(a, )$

b

xy

=

yx

### **Necessary Condition for Local Extreme Values**

If  $f(x, y)$  has a relative extreme value (local maximum or local minimum)

$\partial f$

$\partial f$

at a point  $(a, b)$  and if  $f =$

,  $f =$

exist, then  $f(a, b)$

and

$x$

$= 0$

$x$

$\partial x$

$y$

$\partial y$

$f(a, b)$

.

$y$

$= 0$

### **Second Derivative Test for Local Extreme Values**

Let  $f(x, y)$  possess continuous first and partial derivatives at a critical point

$(a, b)$ . At point  $(a, b)$  we define the quantity



f

f

xx

xy

2

D(a, )

b =

= f f - f

xx yy

xy

f

f

yx

yy

Then,

• If  $D > 0$  and f

, then  $f(a, b)$  is a **local minimum**.

xx > 0

• If  $D > 0$  and f

, then  $f(a, b)$  is a **local maximum**.

xx < 0

• If  $D < 0$  then  $f$  has neither a local minimum nor a local maximum

point at  $(a, b)$ . The point is known as a saddle point.



African Virtual University 83

### Learning Activity

1. The volume of a cube with sides of length  $x$ ,  $y$ ,  $z$  is given by the formula

$V =$

.

$xyz$

2. The volume of a right circular cone of height  $h$  and base radius  $r$  is given

1

by  $V = \frac{1}{3} \pi r^2 h$ .

3

### Mathematical Problem

#### Definitions of Function, Domain and Range

What is common in all two examples given above is that, in each case, a rule is given for assigning a unique value to the respective dependent variable (Area or Volume) whenever a set of the corresponding independent variables is specified.

Typically, if the radius of the circular cone in Example 3 is fixed to be 2 and the

height is 3 , then the volume of the cone has the value  $V = \pi$

4 .

The learner is strongly encouraged to recall the corresponding definitions for a function of a single independent variable and see the close similarity or even outright identity in the definitions.

In general, we may define **a function  $f(x_1, x_2, x_3, \dots, x_n)$  of  $n$  independent variables** as a rule or a formula which assigns a unique number

$z$

$z = f(x_1, x_2, x_3, \dots, x_n)$

where

**$x_1, x_2, x_3, \dots, x_n$  are independent variables** as a rule or a formula which assigns a unique number

$$z = f(x_1, x_2, x_3, \dots, x_n)$$

where

$x_1, x_2, x_3, \dots, x_n$

are independent variables

and  $z$  is a real number

for every given set

$\{x_1, x_2, x_3, \dots, x_n\}$

of real numbers

such that

$x_i \in D$

for every given set  $\{x_1, x_2, x_3, \dots, x_n\}$

where

$D \subset \mathbb{R}$

and

$z \in \mathbb{R}$

Symbolically, such a mapping is represented by writing  $f$  :

$\mathbb{C} \mathbb{R}^n$

$D$

$\rightarrow \mathbb{R} .$

The domain of a function of several independent variables is the set  $D$  of all elements  $\{ x_1, x_2, x_3, \dots, x_n \}$  for which the function  $f$  has a unique value.

1

2

3

$n$

The range of a function of several independent variables is the set of values of the function  $f$  for all points  $\{ x_1, x_2, x_3, \dots, x_n \}$  taken from the domain  $D$  . For 1

2

3

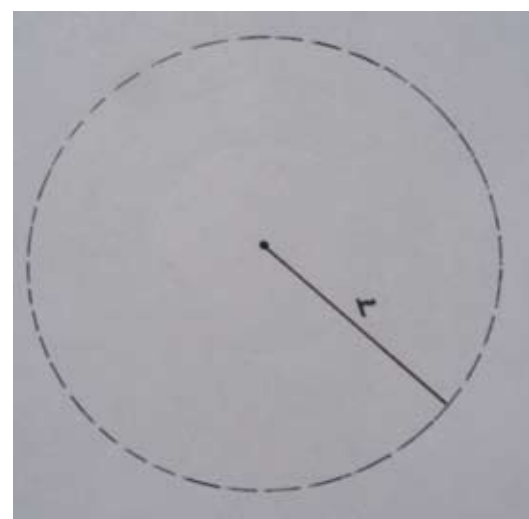
$n$

a function of two independent variables, the range is often a connected surface (roof or cover) spanned above the domain.

Please Note This Carefully

For purposes of maintaining simplicity of presentation and thereby hopefully gain clarity of understanding, but without loss of generality, we shall limit our treatment of functions of several independent variables to functions of two independent variables.





## African Virtual University 84

### Example

Consider the relationship  $z = f(x, y) = \ln(2$

$$2^{x+y-4}).$$

For this relationship

to represent a function of the two independent variables  $(x, y)$  the logarithmic

function must be defined (have a unique value) at such points. This is only possible if

the argument of the logarithm is positive, i.e.,

$2^{x+y-4} > 0$ .

This requirement restricts the choice of pairs  $(x, y)$ , namely to the region or

domain  $D = \{(x, y) \mid$

$$x + y \geq 4\},$$

implying that  $(x, y)$  must lie outside the

open disc  $D = \{(x, y) \mid$

$x + y < 4\}$ .

$$x + y < 4.$$

An “open” disc of radius  $r$

Note: Points lying on the circumference do not belong to the open disc.

The range of this function is the set +

$\mathfrak{R}$  of non-negative real numbers.



African Virtual University 85

Do This Exercise

1. Give as many examples as you can (from real life, mathematics and science in general) of functions of more than one independent variable.

2. For each of the following functions

(i) Find and sketch the domain of the function, and

(ii) Find the range.

(a)

2

2

3

$f(x, y)$

$$f(x, y) = x^2 y + xy^2 - y^3$$

2

$2x^2 y$

(b)  $f(x, y)$

$$f(x, y) =$$

2

2

$1 - x - y$

2

2

(c)

9

$f(,$

$x y)$

$-x - y$

$= e$

(d)  $f(x, y) = \sin(xy)$ .



**African Virtual University 86**

**Level curves**

**One does not need to be a geographer to know that the position of any point on our planet earth is completely determined once its latitude and longitude are known. Equally true is the fact that the altitude (height above mean sea level) of any point depends on its position on the globe.**

**Geographers use contours to obtain a visual impression of the steepness of a given landscape.**

**What are contours?**

**Contours are lines (curves) on a map connecting all points on a given landscape which have the same height above mean sea level.**

The idea of contours has been adopted by mathematicians to gain some insight of the shape of a function of two independent variables. Instead of the word “contour” mathematicians use the more descriptive phrase “level curves”.

What are Level Curves?

A level curve of a given function  $f(x, y)$  is any curve described by an equation of the form  $f(x, y) = C$ , where  $C$  is a parameter that may take any value in the range of  $f$ . In other words, a level curve is the locus of all points  $(x, y)$  in the domain  $D$  of  $f(x, y)$  for which the function has the constant value  $C$

Examples

1. The level curves of the function

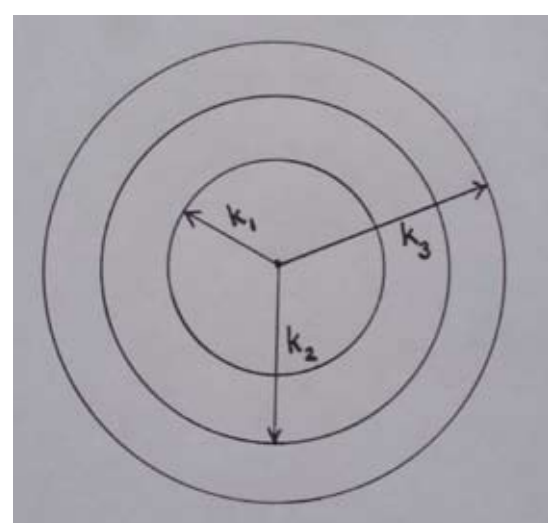
2

2

$f(x, y) = x^2 + y^2$  are circles given by the

equation  $x^2 + y^2 = k$  in which  $k$  is a positive constant. For different choices of the parameter  $k$  the circles are concentric with centre at the origin and have

radius  $r = \sqrt{k}$ .





Concentric circles of radii  $r = k$ ,  $k$  and  $k$ .

1

2

3

Do this

Sketch some level curves for the function  $f(x, y) = 2x + y$ .



Limit of a function of two variables

As pointed out in Unit 1 of this same Module in connection with a function

$f(x)$  of a single variable, the limit concept is also fundamental for the analysis

of functions of several variables. The definition of a limit is given in Section 1.4

as one of the key concepts of this unit.

As is true elsewhere in mathematics, it is one thing to define a concept, such as

that of a limit, and quite another to use it. It is particularly difficult for the limit

concept, in that the definition inherently assumes that one already knows the limit!

Here we shall lean heavily on the first two key theorems on limits listed in Section

1.1: Key Theorems / Principles. We shall discuss three illustrative examples and

let the learner do the same in a group work on some problems.

Example 1 (An application of Theorem 1)

$xy$

[

$xy$  ]

Let  $f(x, y) =$

. We now pose the question: Does

lim

2

2

x

+ y

(x, y) (0,0 ]

2

)

|

→

2

[ x + y ]

exist? In an attempt to find an answer to this question we let point (x, y) approach

(

)

0

,

0

along the line  $y=mx$  where the slope  $m$  of the line is purposely left as a

parameter. In this case

[

xy ]

2

[

mx

]

m

lim

= lim

=

|

|

.

(x, y) (0,0 |

2

)

|

→

2

2

2

2

2

[ x + y ]

x → 0

x + m x

1+ m

[  
]

Since the limit depends on the slope of the line, we conclude that the func-

xy

tion  $f(x, y) =$

has no limit at the origin. This result follows from

2

2

x

+ y

**Theorem 1**



**African Virtual University 89**

**Example 2 (A second application of Theorem 1)**

2

2 xy

Consider now the function  $f(x, y) =$

and pose the same question: Does

2

4

x + y

[

2

$2xy$  ]

limit

lim |

exist? Adopting the same linear approach with

2

|

$(x, y) \rightarrow$

4

$(0,0)$  [  $x + y$  ]

2 2

$m x$

[

2 2

$m$

]

$y = mx$  we get

$f(,$

$x y) =$

$= x$

, and hence conclude

4

2

|

2

2 |

$1 + mx$

$[ 1$

$( + mx ) ]$

$[$

$2$

$2xy ]$

$[ 2 2$

$m$

$]$

that  $\lim$   $|$

$= \lim$

$x = 0$  regardless of the direction one

$2$

$|$

$|$

$|$

$( x, y ) \rightarrow$

$2$

$(0,0)$

$2$

$2$

$[ x + y ]$

$x \rightarrow 0 1$

$[ + mx ]$

chooses in approaching the origin.

This result may tempt one to conclude that zero is the limit of the function at the origin. However, let us be careful. Key Theorem No.1 warns us that we must be very careful in using this approach, for we have not exhausted all possible directions of approach. Indeed we soon find out that if we take the path defined

$$\left[ \begin{array}{l} 2 \\ 2xy \end{array} \right]$$

$$\left[ \begin{array}{l} 2 \\ 4 \\ y \end{array} \right]$$

by the curve

$$x = y \text{ then } \lim = \lim$$

$$= 1$$

$$2$$

$$|$$

$$|$$

$$|$$

$$(x, y) \rightarrow$$

$$4$$

$$(0,0)$$

$$4$$

$$4$$

$$\left[ x + y \right]$$

$$y \rightarrow 0 \lfloor y + y \rfloor$$

This proves that the function has no limit at the origin because we get two different limits depending on the path we take to approach the origin.

What reliable method can we then adopt in trying to determine limits of functions of several (two) variables?

### Example 3 (An application of Theorem 2)

An example taken from Robert T. Smith and Roland B. Minton: **Multivariable**

**Calculus, p. 44.**

2

$x y$

Let  $f(x, y) =$

, and consider

lim

$f(x, y)$

. The learner is

2

2

$x + y$

$(x, y) \rightarrow (0, 0)$

requested to show that, in this case, any approach to point the (

)

0

,

0

along any



straight line  $y = mx$  will give lead to the limit

$L = 0$ . This is equally true

if one takes the paths

2

$y = x$  or

2

$x = y$ . We may therefore reasonably suspect

2

$x y$

that  $\lim$

$=$ .

0 We consider applying Theorem 2 to prove that indeed

2

2

$(x, y) \rightarrow (0,0)$   $x + y$

the limit of the function is zero as  $(x, y) \rightarrow (0,0)$ .



African Virtual University 90

We begin by noting that since for all  $(x, y) : x^2 + y^2 \geq x^2$ , it follows that

$x^2 y$

$x^2 y$

$$f(x, y) - L =$$

$\leq$

$= y$ . The function  $g(x, y)$  sighted in

$$x^2 + y^2$$

$x^2$

**Theorem 2 is in this case**

$g(x, y) = y$ . But  $\lim g(x, y) = 0$ .

Therefore,

$$(x, y) \rightarrow (0, 0)$$

$2$

$x y$

**by Theorem 2, it follows that  $\lim$**

$=$

$0$

$2$

$2$

$$(x, y) \rightarrow (0, 0) \quad x + y$$

**Do This**

**Using the above three examples and working with a colleague whenever and wherever possible, determine whether or not the following limits exist. Where a limit exists, find it.**

**1.**

**$\lim$**

**(2**

2

$x + y - 4$ )

$(x, y) \rightarrow (0, 0)$

$[x - y]$

2.

lim

$(x, y) \rightarrow (0, 0)$

2

)

|

$\rightarrow$

2

$[x + y]$

$[x - 2$

)

$1 \ln(x)]$

3.

lim

|

Consider limits as  $(x, y) \rightarrow$  ,

1

(0) along

2

|

$(x, y) \rightarrow$

2

(1,0 )

[ (x - )

1 + y ]

the three: along  $x = 1$ ; along  $y = 0$  , and along  $y = x - 1$  . Do you suspect anything?

### Continuity of a function of two variables

In Unit 1 of this Module we came across the concept of continuity of a function of one independent variable on two occasions.

The first time was in Section 1.4 on Key Concepts where the concept of continuity at a point  $x = a$  was first defined. At that juncture we listed three necessary and sufficient conditions for a function  $f ( x )$  to be continuous at a point  $x = a$  namely:

(i) The function  $f$  must have a value (defined) at the point, meaning that  $f ( a )$  must exist;

(ii) The function  $f$  must have a limit as  $x$  approaches  $a$  from right or left, meaning that  $\lim f ( x )$  must exist (let the limit be  $L$  ); and

$x \rightarrow a$



African Virtual University 91

(iii) The value of the function at  $x = a$  and the limit at  $x = a$  must be equal, meaning  $L = f ( a )$  .

In short, we summed up these conditions with the single equation  $\lim f ( x ) = f ( a )$

$x \rightarrow a$

The second time we came across the concept of continuity was in Section 1.7.2, where we learnt how to determine whether or not a given function  $f(x, y)$  is continuous at a point or over an entire interval  $a \leq x \leq b$ , including what was referred to as removable discontinuity.

In the current unit (Unit 4) we are dealing with functions of more than one independent variable. Specifically, we are considering a function  $f$  of two independent variables:  $w = f(x, y)$

We want to define the concept of continuity for a function of two independent variables. Again we note that we already have defined the concept in Section 1.4 of this Unit. The definition is exactly the same as that for a function of a single variable, namely,  $f(x, y)$  is said to be continuous at a point  $(a, b)$  in its domain if, and only if

$\lim$

$f(x, y)$

$(x, y) \rightarrow (a, b)$

$(x, y) \rightarrow (a, b)$

$(x, y) \rightarrow (a, b)$

(i) If either  $f(x, y)$  is not defined at point  $(a, b)$  or

(ii) If

$\lim$

$f(x, y)$  does not exist, or

$(x, y) \rightarrow (a, b)$

(iii) If

$\lim$

$f(x, y)$  exists but is different from  $f(a, b)$

$(x, y) \rightarrow (a, b)$

then we can conclude that  $f(x, y)$  is not continuous (discontinuous) at  $(a, b)$ .

In the special case where the function is defined at the point and the limit exists, but the limit and the function value are different, one speaks of a removable discontinuity, for one can under such circumstances define a new function

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \neq (a, b) \\ \lim_{(x,y) \rightarrow (a,b)} f(x, y) & \text{if } (x, y) = (a, b) \end{cases}$$



African Virtual University 92

that clearly satisfies all the three stated conditions for continuity at  $(a, b)$ .

### Examples

We wish to discuss continuity of each of the following three functions at the points given. The three examples have purposely been chosen to demonstrate the three possible situations: continuity, removable discontinuity and non-removable discontinuity.

1.  $f(x, y) = xy^2 - 2x^2y + 3x$  at  $P( )$

1

,

2 .

We observe that  $f$  is a polynomial of degree two in two variables. A polynomial function is continuous everywhere, and therefore it is continuous at (

)

1

,

2 .

Specifically we note that:

• The function is defined at the point:

$f( )$

1

,

2

= 0

• The function has a limit at the point:  $\lim ( 2$

$xy - 2^2$

$x y + 3x) = 0$

$(x, y) \rightarrow (2)$

1

,

• The limit is equal to the value of the function at the point. Therefore, the

function is continuous at  $(2,1)$

[1 y 2

+

]sin( x)

2. f ( x, y) =

at P(

)

0

,

0

.

x

We note that this function is not defined at point: f (

)

0

,

0

does not exist, and

therefore, the function is not continuous at the origin. However, we note that the

function has a limit at the origin, since

[1 2

+ y ]sin(x)

x

lim

[

= lim(1



2

+ y)  $\lim$

sin()

lim

$\lim_{(x,y) \rightarrow (0,0)} = 1 \times 1 = 1$

$(x, y) \rightarrow (0,0)$

x

$| y \rightarrow 0$

$|$

$\lim_{x \rightarrow 0}$

$|$

$\lim$

x  $\lim$

Because the limit exists but the function is discontinuous, we

have a typical case of a removable discontinuity. The derived function

$f(x, y)$

if  $(x, y) \neq (0, 0)$

$(x, y) \neq (0, 0)$

0

,

0

$|$

F(x, y)

$(x, y) = \begin{cases} \dots \end{cases}$

is continuous at (0, 0)

)

0

,

0

.

| l 1

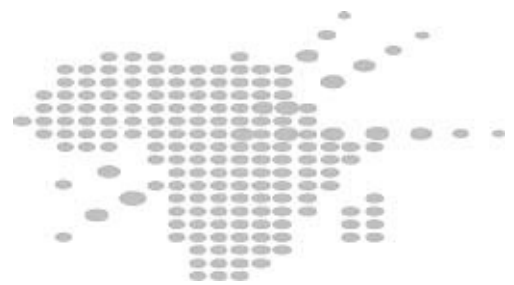
if (,

x y) = ( )

0

,

0



**African Virtual University 93**

x

3.  $f(x, y) =$

at  $P($

)

0

,

0

.

2

2

$$x + y$$

Clearly, this function is not defined at the origin and therefore, it is discontinuous at the point. As to whether the function has a limit at the origin, we note the following:

If we approach the point  $(0,0)$  along the along the y-axis ( $x = 0$ ) we see that the function is identically zero and hence it has zero limit.

If, on the other hand, one approaches the origin along the x-axis ( $y = 0$ ) we note that

1

the function takes the form  $f(x, y) = \frac{y}{x}$ , which has no limit as  $x \rightarrow 0$ . We

x

therefore conclude that the function has no limit and hence is not continuous at

(

)

0

,

0

. In this case the discontinuity is not removable.

**Now Do This**

**Discuss the continuity of the following three functions at the points given. If**

**the function is discontinuous, state giving reasons, whether the discontinuity is removable or not removable.**

2

$$xy - y$$

1.  $f(x, y) =$

$$x y) =$$

;(

)

1

,

2

2

$1 - x - y$

y

2.  $f(x, y) =$

;(

)

0

,

0

$x - y$

$1 - x + y$

3.  $f(x, y) =$

;(

)

0

,

0

2

2

$x + y$

## Partial Differentiation

One of the key concepts covered in the second learning activity of Unit one is the derivative of a function  $f(x)$  of a single independent variable. In Section 1.4 of this Unit a key concept of partial derivatives of a function  $f(x_1, x_2, x_3, \dots, x_n)$

2

3

$n$

of several independent variables is mentioned. We now make a formal definition of this important concept specifically in connection with functions  $f(x, y)$  of two independent variables.



African Virtual University 94

### Partial Derivatives

Let  $w = f(x, y)$  be a function of the two variables  $(x, y)$  in a domain  $D$

$D \subset \mathbb{R}^2$

of the  $xy$ -plane and  $(a, b)$  be a point in  $D$ . The function  $g(x) = f(x, b)$ , obtained from  $f(x, y)$  by holding  $y$  at the constant value  $y = b$  is a function of  $x$  alone and is defined over an interval on the  $x$ -axis that contains the value

$x = a$ . One may therefore wish to find out whether  $g$  is differentiable at  $x = a$ . This involves finding

$$\frac{g(a+h) - g(a)}{h}$$

$$\frac{f(a+h, b) - f(a, b)}{h}$$

[

-

]

lim

= lim

→

h 0 |

|

→

h 0 |

|

]

h

]

]

h

]

If the limit exists it is called the partial derivative of  $f(x, y)$  with respect to the variable  $x$  at point  $(a, b)$  and is denoted by

$\frac{\partial f}{\partial x}$

$\frac{\partial w}{\partial x}$

$(a, b)$

or  $f_x(a, b)$  or

$\frac{\partial f}{\partial x}$

$x$

$\frac{\partial f}{\partial x}$

**f**

$\partial$

**The partial derivative**

**(a, )**

**b is defined similarly by holding x constant at**

**y**

$\partial$

**the value  $x = a$  and differentiate the function  $h(y) = f(a, y)$  at  $y = b$  .**

**If we allow the point  $(a, b)$  to vary within the domain  $D$  then the resulting partial derivatives will also be functions of the variable point  $(x, y)$  . We therefore have**

$$\left[ f(x + h, y) - f(x, y) \right]$$

**f**

**lim**

$\partial$

**=**

**(,**

**x y)**

**;**

**$h \rightarrow 0$  |**

**[**

**h**

**]]**

**x**

$\partial$

**[ f (,**

**x y + k) - f (,**

**x y)]**

**f**

**lim**

$\partial$

**=**

**(,**

**x y) .**

**k → 0 |**

**[**

**k**

**]**

**y**

$\partial$



**African Virtual University 95**

**Worked Example**

**Find the partial derivatives of the function**

**3**

**2**



2

3

$$f(x, y) = 2xy + 3xy + 2xy$$

By considering  $y$  as a constant and differentiating  $f$  as if it were only a

$f$

$\partial$

function of the variable  $x$  we get

2

2

3

(,

$$xy) = 6xy + 6xy + 2y$$

.

$x$

$\partial$

Similarly, by considering  $x$  as a constant and differentiating  $f$  as if it were

$f$

$\partial$

only a function of the variable  $y$  one gets

3

2

2

(,

$$xy) = 2x + 6xy + 6xy.$$

$x$

$\partial$

**Now Do This**

**Find the partial derivatives of the following functions:**

[ 2

x ]

1.  $f(x, y) = \ln | 3 |$

[ y ]

$\sin( 2\theta )$

2.  $f(r, \theta) =$

2

r



**African Virtual University 96**

**Higher Order Partial Derivatives**

$\partial f$

f

$\partial$

**We have seen that the partial derivatives**

**and**

**of a function  $f(x, y)$**

$\partial x$

y

$\partial$

of two independent variables are in turn functions of the same two independent variables. Therefore, one can consider their partial derivatives in the same way we did for the original function. One may therefore attempt to find the derivatives

$$\frac{\partial}{\partial x} \left[ \frac{\partial f}{\partial y} \right]$$

$$\frac{\partial}{\partial y} \left[ \frac{\partial f}{\partial x} \right]$$

$$\frac{\partial}{\partial x} \left[ \frac{\partial f}{\partial x} \right]$$

$$\frac{\partial}{\partial y} \left[ \frac{\partial f}{\partial y} \right]$$

|

|;

;

;

.

$\frac{\partial}{\partial x}$

|

|

|

|

|

|

$$\left[ \frac{\partial}{\partial x} \right]$$

$$\frac{\partial}{\partial x} \left[ \frac{\partial f}{\partial y} \right] \frac{\partial}{\partial x} \left[ \frac{\partial f}{\partial y} \right] \frac{\partial}{\partial y} \left[ \frac{\partial f}{\partial y} \right]$$

If limits of the associated difference quotients exist, then we call them the second partial derivatives of  $f$  and denote them by

$$\frac{\partial}{\partial x} \left[ \frac{\partial f}{\partial y} \right]$$

$\partial f$

$\partial [ \partial f ]$

$\partial^2 f$

|

| =

or f ;

=

or f ;

$\partial x$

xx

xy

[  $\partial x$  ]

2

x

$\partial$

|

|

$\partial y [ \partial x ]$

$\partial y \partial x$

$\partial [ \partial f ]$

$\partial^2 f$

$\partial [ \partial f ]$

2

$\partial f$

|

| =

or f ;

=

or f .

$\partial x$

$y x$

|

|

2

$y y$

$\left[ \frac{\partial y}{\partial x} \right] \frac{\partial x}{\partial y}$

$\frac{\partial y}{\partial x} \left[ \frac{\partial y}{\partial x} \right]$

y

$\partial$

**Note Carefully**

**The learner should note carefully the notation used for the two mixed second**

$\partial^2 f$

$\partial^2 f$

$\partial^2 f$

**partial derivatives**

**and**

**. The notation**

**or f means that**

$\frac{\partial y}{\partial x}$

$\frac{\partial x}{\partial y}$

$\partial y \partial x$

$xy$

one differentiates first with respect to  $x$  and then with respect to  $y$ , while

$\partial^2 f$  or  $f$  means differentiation first with respect to  $y$  and then with

$\partial x \partial y$

$yx$

respect to  $x$ .

### Worked Example

Find all four second partial derivatives of the function  $f(x, y) = y \cos(xy)$ .

One finds the following results



### African Virtual University 97

$\partial f$

$= f =$

$2$

$- y \sin(xy) ;$

$\partial x$

$x$

$f$

$\partial$

$= f = \cos()$

$xy - x y \sin()$

$xy ;$

**y**

**$\partial$**

**y**

**2**

**$\partial f$**

**= f =**

**3**

**- y cos( )**

**xy ;**

**2**

**x**

**$\partial$**

**xx**

**$\partial^2 f$**

**= f = - 2ysin( )**

**2**

**xy - xy cos( )**

**xy ;**

**$\partial y \partial x$**

**xy**

**$\partial^2 f$**

**= f = - xsin( )**

**xy - ysin( )**

**2**

**xy - xy cos( )**

$xy$  ;

$\frac{\partial^2 f}{\partial x \partial y}$

$yx$

2

$\frac{\partial f}{\partial x}$

$= f = -2x \sin(x)$

2

$xy - x y \cos(x)$

$xy$

2

$y$

$\frac{\partial}{\partial x}$

$yy$

**Now Do This**

**Find the second partial derivatives of the following functions**

1.  $f(x, t) =$

3

2

2

3

$2xy + 3xy + 4xy.$

2.  $f(x, y) = \ln(23$

$xy).$

**Equality of Second Mixed Partial Derivatives**

**From the above worked example and from the results the Learner will get by**

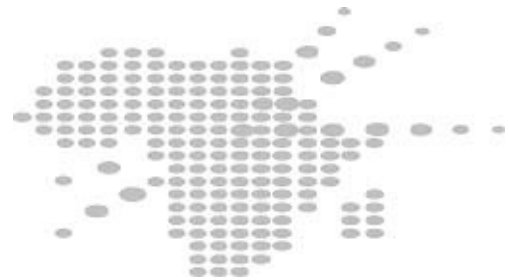


solving the last two self exercise problems, do you detect any relationship between the two mixed partial derivatives  $f_{xy}$  and  $f_{yx}$ ? State the relationship and look

$f_{xy}$

$f_{yx}$

back at the key theorem stated in item 5 of Section 1.5 on Key Theorems and/or Principles.



African Virtual University 98

What we find is that in all cases the mixed second partial derivatives are equal.

This result is not by accident but a well established result based on the theorem referred to and which states that if  $f_{xy}$  and  $f_{yx}$  are both continuous over the domain

$f_{xy}$

$f_{yx}$

$D \subset \mathbb{R}^2$

of  $f(x, y)$  then  $f_{xy}$  and  $f_{yx}$  are equal at all points  $(x, y) \in D$ .

$f_{xy}$

$f_{yx}$

Clearly, partial derivatives of order higher than two can be computed in exactly the same way by differentiating second partial derivatives. This will yield eight third partial derivatives, namely

$f_{xxy}$ ,  $f_{xyx}$

$f_{xyx}$ ,  $f_{xxy}$

$f_{xyx}$ ,  $f_{xxy}$

$f$

$f$

$f$

.

xxx

xyy

xyx

xyy

yxx

yxy

yyx

yyy

**Do this**

**1. Find all eight third partial derivatives of the function  $f(x, y) = \cos(2x + 3y)$  note the equality of the following sets of mixed third partial derivatives:**

$[f, f, f;$

$f, f, f.$

xyy

yxy

yyx ]

xyy

xyx

yxx ]

**2. Show that the function  $f(x, y) = e^{-x} \cos(y)$  satisfies the equation  $f$**

$f$

.

$x +$

$y$

$= 0$

$y$

3. Find the second partial derivatives of the function  $f(x, y) = \int -t$

te dt

$x$

.

### Partial Increments, Total Increment and Total Differential

#### Partial Increments

$\partial f$

$f$

$\partial$

In deriving the partial derivatives

and

of a function  $z = f(x, y)$  of two

$\partial x$

$y$

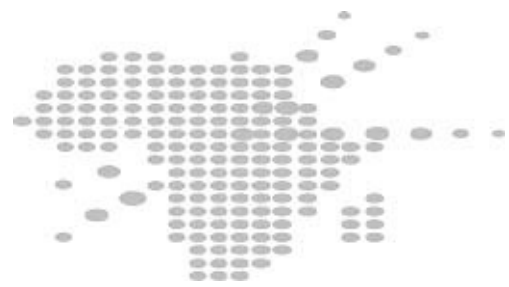
$\partial$

independent variables we kept one of the variables constant while giving the other

$\partial f$

variable a small increase. Specifically, in the case of the partial derivative

$\partial x$



## African Virtual University 99

we kept  $y$  constant and gave  $x$  a small increment  $\Delta x$ . We call the corresponding change in the value of the function  $f(x + \Delta$ ,

$x, y) - f(x,$

$x, y)$  the partial

increment of  $z$  with respect to  $x$ , and denote it by

$\Delta z$

$f(x + \Delta,$

$x, y) - f(x,$

$x, y$

$x$

$=$

)

Similarly, one defines the partial increment of  $z$  with respect to  $y$ , written

as

$f(x + \Delta,$

$x$

$y + \Delta$

$+ z, y$

$\Delta) - f(x,$

$x, y$

$+ +$

$\Delta$ ,

$x y$

$\Delta$ )

$y \pm f (,$

$x y +$

)

$y$

$\Delta) - f (,$

$x y)$

$y$

=

**The quantities  $\Delta z$  and  $\Delta z$  are called partial increments because, in each case,**

$x$

$y$

**only one of the variables is allowed to change, the other being kept constant.**

**Total Increment**

**If we allow both variables  $x$  and  $y$  to change by small amounts  $\Delta x$  and**

**$\Delta y$  respectively, then the change  $f(x + \Delta x, y + \Delta y) - f(x, y)$  in the function**

**value is called the total increment in  $z$  with respect to both its variables and is denoted by**

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

**Do this**

**Find the partial increments  $\Delta z$  and  $\Delta z$  of the function  $z$**

$x^2$

=

**$y$  and use**

$x$

$y$

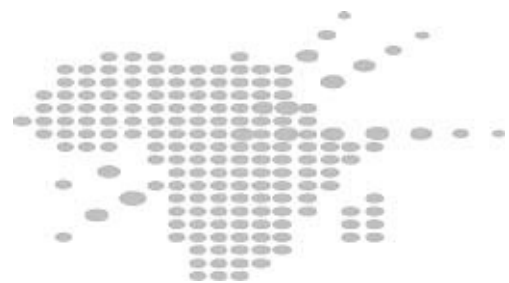
the results to show that, in general,  $\Delta z \neq \Delta z$

$\Delta$

$x$

$y$

$\neq z$



African Virtual University 100

### Total Differential

$\partial f$

Let us attempt to relate the total increment with the partial derivatives

$\partial x$

$f$

$\partial$

and

at  $(x, y)$ . By adding and subtracting the term  $f(x, y + \Delta y)$  we can

$y$

$\partial$

express  $\Delta z$  in the form

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

$$= f(x + \Delta x, y) + f(x, y + \Delta y) - f(x, y) - f(x, y)$$

$$= f(x + \Delta x, y) - f(x, y) + f(x, y + \Delta y) - f(x, y)$$

$\Delta) - f(,$

$x y + y$

$\Delta) + f(,$

$x y + y$

$\Delta) - f(,$

$x y)$

$[ f(x + x$

$\Delta, y + y$

$\Delta) - f(x, y + y$

$\Delta)$

$[ f(x, y + y$

$\Delta) - f(x, y)]$

=

$]x$

$\Delta$

|

+

y

$\Delta$

|

|

L

x

|

$\Delta$

]

y

[

Δ

]

f

∂

f

∂

=

(x + ξ x

Δ, y + y

Δ) x

Δ +

(x, y + η y

Δ) y

Δ

x

∂

y

∂

where the quantities ξ and η lie in the interval ( )

1

,

0.



If we now assume that the function  $z = f(x, y)$  has continuous partial derivatives, then

$f$

$\partial$

$f$

$\lim$

$(x + \Delta$

$\xi,$

$x y + y$

$\Delta)$

$\partial$

$=$

$(,$

$x y)$  and

$x$

$\Delta \rightarrow 0, y$

$\Delta \rightarrow 0$

$x$

$\partial$

$x$

$\partial$

$f$

$\partial$

$f$

$\lim$

$(,$

$$\mathbf{x} \mathbf{y} + \boldsymbol{\eta} \mathbf{y}$$

$$\Delta)$$

$$\partial$$

$$=$$

$$(\mathbf{,}$$

$$\mathbf{x} \mathbf{y})$$

$$\mathbf{x}$$

$$\Delta \rightarrow \mathbf{0}, \mathbf{y}$$

$$\Delta \rightarrow \mathbf{0}$$

$$\mathbf{x}$$

$$\partial$$

$$\mathbf{y}$$

$$\partial$$

**We can therefore express the total increment in terms of the partial derivatives**

**in the form**

$$\left[ \mathbf{f}$$

$$\partial$$

$$\left[ \mathbf{f}$$

$$\partial$$

$$\right]$$

$$\right]$$

$$\Delta \mathbf{z} =$$

$$(\mathbf{x}, \mathbf{y}) + \boldsymbol{\varepsilon}$$

$$+$$

$$(\mathbf{x}, \mathbf{y}) + \boldsymbol{\varepsilon}$$

2

y

1

x

$\Delta$

|

$\Delta$

|

|

$\lfloor x$

|

$\partial$

$\rfloor$

y

$\lfloor \partial$

$\rfloor$



**African Virtual University 101**

$\partial f$

**f**

$\partial$

=

$(x, y)\Delta x +$

$(x, y)$   $y$

$\Delta + \epsilon$

$+ \epsilon^2 \Delta y$

$1 \Delta x$

$\partial x$

$y$

$\partial$

where the quantities  $\epsilon$  and  $\epsilon^2$  tend to zero as  $\Delta x$  and  $\Delta y$  approach zero.

$1$

$2$

$\partial f$

$f$

$\partial$

The term

$(x, y) \Delta x +$

$(x, y) y$

$\Delta$  in the expression for the above total

$\partial x$

$y$

$\partial$

increment and which is linear in  $\Delta x$  and  $\Delta y$  is called the total differential

of  $z$  and is denoted by

$\partial f$

$f$

$\partial$

$dz =$

$(x, y)\Delta x +$

$(x, y) y$

$\Delta .$

$\partial x$

$y$

$\partial$

### Example

Calculate the total differential and the total increment of the function

$$z = f(x, y = x^2$$

)

at point (

)

3

,

2

given that  $\Delta x = 0.2$  and  $\Delta y = 0.1$ .

### Solution

The expressions for the total increment and the total differential are:

$z$

$$\Delta = 2xy \Delta x$$

$$\Delta + x^2 \Delta y$$

$$\Delta + 2x \Delta x$$

$$\Delta y$$

$$\Delta + y(\Delta x$$

$$\Delta)^2 + (x$$

$$\Delta)^2 y$$

$$\Delta$$

$$2$$

$$dx = 2xy x$$

$$\Delta + x y$$

$$\Delta$$

Substituting the values  $x = 2$ ,  $y = 3$ ,  $\Delta x = 0.2$  and  $\Delta y = 0.1$  into both expressions gives

$$dz = .$$

$$2.8 \text{ and}$$

$$\Delta z = 3.004 .$$

We observe from these results that the total differential gives a very good approximation of the total increment of a function. In the example we have solved, the difference between  $dz$  and  $\Delta z$  is less than 8% .



African Virtual University 102

Now Do This

Using the total differential, approximate the total increment in calculating the volume of material needed to make a cylindrical can of radius  $R = 2$  cm, height  $H = 10$  cm and whose walls and bottom are of thickness  $\Delta R = \Delta H = 0.1$  cm.

Chain Rule for Differentiation

The variables  $x$  and  $y$  in the function  $z = f(x, y)$  may themselves be functions of either a single variable  $s$  or of two variables  $s$  and  $t$ .

**The case of a single variable**

If  $x = x(s)$  and  $y = y(s)$  then

$$z = f(x, y) = f(x(s), y(s))$$

is implicitly a function of the single variable  $s$  and therefore one may consider

$df$

finding the ordinary derivative of  $z$  with respect to  $s$ , that is, finding

.

$ds$

The expression for the derivative is found using the chain rule for differentiating a composite function (a function of a function):

$dz$

$\frac{\partial f}{\partial x} dx$

$f$

$\frac{\partial}{\partial y} dy$

=

+

.

$ds \frac{\partial x}{\partial s} ds$

$y$

$\frac{\partial}{\partial s} ds$

Note carefully the different notations used in this expression, partial differentiation of  $f(x, y)$  and the ordinary derivatives of both  $x$  and  $y$ .

**The case of two variables**

If  $x = x(s, t)$  and  $y = y(s, t)$  then

$z = f(x, y) = f(x(s, t), y(s, t))$  is implicitly a function of the two variables  $s$  and  $t$ . One can therefore consider

$\partial f$

$\partial f$

finding the partial derivatives

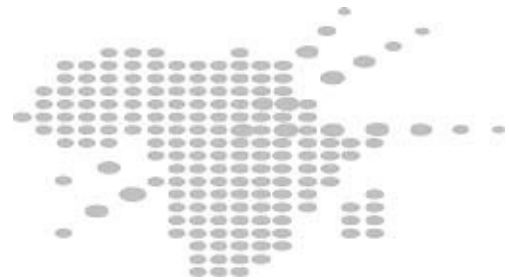
and

. Again, these are found by the chain

$\partial s$

$\partial t$

rule, but this time, partial derivatives are involved throughout.



African Virtual University 103

$\partial z$

$\partial f \partial x$

$\partial f \partial y$

$\partial z$

$\partial f \partial x$

$\partial f \partial y$

=

+

;

=

+



.

$\partial s$

$\partial x \partial s$

$\partial y \partial s$

$\partial t$

$\partial x \partial t$

$\partial y \partial t$

### Example

Given the function  $z = f(x, y) = \ln(x + y)$  where  $x = u^2$

$=$

$v$  and

$2$

$y = uv$ ,

$\partial z$

$\partial z$

express the partial derivatives

and

as functions of  $u$  and  $v$ .

$\partial u$

$\partial v$

### Solution

Direct application of the second set of equations for the chain rule gives

$\partial z$

$2uv$

$v^2$

2

$2uv + v$

$2 + v$

=

+

=

=

$\partial u$

$x + y$

$x + y$

2

2

$u v + uv$

$u + v$

$\partial z$

$u^2$

$2uv$

2

$u + 2uv$

$u + 2$

=

+

=

=

.

$\partial v$

$x + y$

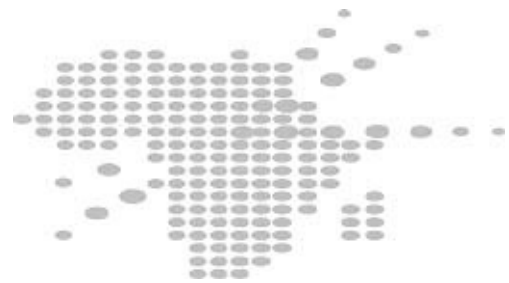
$x + y$

2

2

$u v + uv$

$u + v$



**African Virtual University 104**

**Now Solve**

**dz**

**1. If  $z = f(x, y) =$**

**3**

**2**

**2**

**3**

**$2x y + 3x y + 4xy$  and  $x = \cos \theta$  ,find**

**.**

**$\theta$**

**d**

**2. If  $f(x, y) = 2$**

**2**

**$x + y$  , where and  $x = r \cos(\theta)$  ) find the partial derivatives**

$\partial z$

$\partial z$

and

.

$\partial r$

$\partial \theta$

## **Exercise Questions (From Gill: The Calculus Bible)**

### **Do This**

**Solve all the problems given in**

**Exercise 2.1 (pp 60-61)**

**Exercise 2.2 (pp 71-72)**



**African Virtual University 105**

## **Activity 2: Applications of partial derivatives (19 Hrs)**

### **Specific Learning Objectives**

**At the end of this activity the Learner should be able to:**

- **State Taylor's theorem for a function of two variables**
- **Write down the first three terms of a Taylor's series expansion**
- **Find the coordinates of the center of gravity (mass) of a plane lamina**
- **Define and classify critical points for a function of two variables**
- **Determine local (relative) maxima, minima and saddle points**
- **Use Lagrange multipliers in constrained optimization**
- **Determine coordinates of center of gravity (mass)**

## **Summary**

**This learning activity is devoted to some aspects of applications of preceding calculus (limits, continuity and partial differentiation) of a function of two independent variables. Faced with a long list of possible areas of application we have been forced to select only a few. We have limited our presentation to four areas of application, namely**

- Center of gravity**
- Taylor's theorem**
- Local maxima, minima and saddle points**
- Lagrange multipliers for solving constrained optimization problems.**

**Presentation of these areas of application follows our familiar pattern of giving a short theoretical note, illustration of such theory on an example or two followed by at least two self exercise questions to be solved by the learner, alone or working with colleagues.**

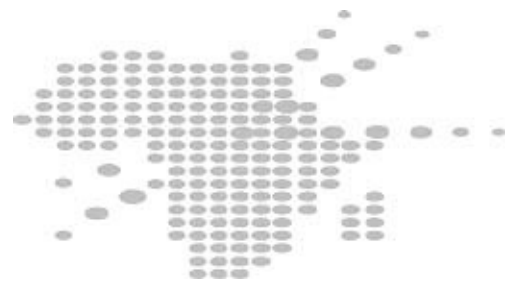
## **Compulsory Reading**

**All of the readings for the module come from Open Source text books. This means that the authors have made them available for any to use them without charge.**

**We have provided complete copies of these texts on the CD accompanying this course.**

**Multivariable Calculus, George Cain and James Herod**

# Chapter 7 and 8



**African Virtual University 106**

**Internet and Software Resources**

**Wolfram Mathworld (Visited 07.11.06)**

**<http://mathworld.wolfram.com/>**

**Search for: Center of gravity, Taylor's theorem, Local maxima, Local minima, Saddle points, Lagrange multipliers [If you are using this document on a computer, then the links can be clicked directly]**

**Key Concepts**

**Center of gravity**

**The center of gravity is also referred to as the center of mass. This is a point at which vertical and horizontal moments of a given system balance.**

**Taylor's formula**

**This seeks to extend the Taylor series expansion of a function  $f(x)$  of a single variable at a point  $x = a$  to a function  $f(x, y)$  of two variables at a point**

**$(a, )$ .**

**$b$**

**Relative Extrema**

**Relative extrema is a collective terminology for the relative maximum and minimum values of a function. The singular form of the word is relative extremum, which may be a relative maximum or a relative minimum value.**

# Lagrange Multipliers

Lagrange multipliers are the numbers (or parameters) associated with a method known as the Lagrange multipliers method for solving problems of optimization (extrema) subject to a given set of constraints.



African Virtual University 107

## Key Theorems and/or Principles

### Necessary Condition for Local Extreme Values

If  $f(x, y)$  has a relative extreme value (local maximum or local minimum) at

$\partial f$

$\partial f$

a point  $(a, )$

$b$  and if

$f =$

,  $f =$

exist, then  $f(a, )$

$b$

$x$

$= 0$

$x$

$\partial x$

$y$

$\partial y$

and  $f(a, b)$

$b$

.

$y$

$= 0$

### Second Derivative Test for Local Extreme Values

Let  $f(x, y)$  possess continuous first and partial derivatives at a critical point

$(a, b)$

$b$ . At point  $(a, b)$

$b$  we define the quantity

$f$

$f$

$xx$

$xy$

$2$

$D(a, b)$

$b =$

$= f f - f$

.

$xx yy$

$xy$

$f$

$f$

$yx$

$yy$



**Then,**

**• If  $D > 0$  and  $f$**

**,**

**then  $f(a, b)$**

**$(a, b)$  is a local minimum.**

**$D_{xx} > 0$**

**• If  $D > 0$  and  $f$**

**,**

**then  $f(a, b)$**

**$(a, b)$  is a local maximum.**

**$D_{xx} < 0$**

**• If  $D < 0$  then  $f$  has neither a local minimum nor a local maximum point**

**at  $(a, b)$**

**$(a, b)$  . The point is known as a saddle point.**

## **Learning Activity**

### **Center of Gravity**

**In this module we will limit our presentation of the concept of center of gravity to functions of two variables.**

**Let us consider a thin, flat plate (a lamina) in the shape of a region  $D$  in  $\mathbb{R}^2$ ,**

**$\mathbb{R}^2$ ,**

**whose surface density at any point  $P \in D$  is a function  $\rho(x, y)$  of the coordi-**

**nates  $(x, y)$  of the point. Our interest is to find the coordinates  $x$  and  $y$  of**

**the point  $C \in D$  where, from an engineering point of view, the plate will balance if a support is placed there.**



## African Virtual University 108

The point  $C(x, y)$  is called the center of gravity or the center of mass of the lamina.

Calculation of the coordinates  $x$  and  $y$  of the center of mass involves calculating three related quantities, namely:

- The total mass  $M$  of the region  $D$  ;
- The vertical moment  $M_x$  of the lamina about the  $x$  – axis.  $M_x$  is a measure

$x$

$x$

of the tendency of the lamina to rotate about the  $x$  – axis.

- The horizontal moment  $M_y$  of the lamina about the  $y$  – axis.  $M_y$  is a

$y$

$y$

measure of the tendency of the lamina to rotate about the  $y$  – axis.

After calculating these three quantities, the coordinates of the center of gravity are then obtained using the formulas:

$M_x$

$M_y$

$x =$

$y ;$

$y =$

$x .$

M

M

### Calculation of the Total Mass M

The total mass M of a region D in the xy – plane and having surface density function (mass per unit area)  $\rho(x, y)$  is given by the value of the double integral

$$M = \iint_D \rho(x, y) dx dy.$$

D

### Example

If the mass of a circular plate of radius r is proportional to its distance from the center of the circle, find the total mass of the plate.

### Solution

The surface density of the plate is indirectly given as

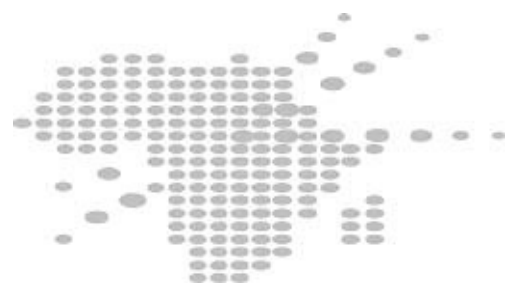
2

2

$\rho(x,$

$y) = k(x + y)$  where

k is a constant of proportionality. The total mass is then calculated from



African Virtual University 109

r

$[2\pi^2$

]

2

$M =$

$k x^2 y \, dx \, dy$

$\iint$

$+ 2$

$= \int$

$r \, d\theta \, dr$

$\int$

$\int$

$=$

3

$k\pi r^3$

3

D

0

$\int_0$

$\int$

**Calculation of the Vertical Moment  $M_x$**

**The vertical moment of the lamina is given by  $M =$**

**$\rho(x, y) y \, dx \, dy$**

**x**

**$\iint_D$**

**Calculation of the Horizontal Moment  $M_y$**

**The vertical moment of the lamina is given by  $M =$**

**$\rho(x, y) x \, dx \, dy$**

y

$\iint_D$

**Example**

**Determine the coordinates of the center of gravity of the section of the ellipse**

$2$

$2$

**x**

**y**

**+**

**= 1 which lies in the first quadrant, assuming that the surface density**

$2$

$2$

**a**

**b**

$\rho(,$

**x y) = 1 at all points.**

**Solution**

**The total mass**

$2$

$2$

**b a**

**a**

**x**

**[**

**-**

]

$$M = \iint dx dy = \int dy \int dx$$

a

f

D

0

0

|

L

]

Using the substitution  $x = a \cos \theta$  the above iterated integral gives the value

1

of the total mass as  $M = ab\pi$

4

Similarly, using the same substitution, the values of the horizontal and vertical moments



African Virtual University 110

b a a<sup>2</sup> x

[ - 2 ]

M =

x dx dy =

|

$dy | x dx$  and

$y$

$\int$

$a f$

$\int$

$D$

$0 |$

$0$

$|$

$L$

$J$

$2$

$2$

$b a$

$a$

$x$

$[$

$-$

$]$

$M =$

$ydxdy =$

$|$

$dy | y dx$

$x$

$\int$

**a f**

**f**

**D**

**0 |**

**0**

**|**

**L**

**J**

**2**

**a b**

**2**

**ab**

**are found to be M =**

**and M =**

**y**

**3**

**x**

**3**

**4a**

**4b**

**These results lead to the coordinates x =**

**and y =**

**for the center**

**$\pi$**

**3**



$\pi$

3

of gravity of lamina.

**Self Exercise**

1. Assuming a constant surface density  $\rho(x, y) = k$  find the coordinates of the center of mass of the upper half of a circular lamina of radius  $r$ .

2. Find the center of gravity of a thin homogeneous plate covering the region  $D$  enclosed between the  $x$  - axis and the curve

2

$x = 2y - y^2$ .



**African Virtual University 111**

**Taylor's Formula**

On page 354 of our basic reference: *The Calculus Bible* by Brigham Young we encountered Taylor's formula for a function  $y = f(x)$  of a single variable about a point  $x = a$ . The relevant theorem (Theorem 8.7.1) states that, if  $f$  and all its derivatives'

,

''

( $n+1$

$f, f', f'', \dots, f^{(n)}$

are continuous for all  $x$  in a small neighborhood

of a point  $x = a$ , then

$n + 1$

$$f(x) = \sum_{k=0}^n (x - a)^k \frac{f^{(k)}(a)}{k!} + R_{n+1}$$

$f$

$(a) + \mathbb{R}^n$

$k=0 !$

$k$

$1$

where  $R =$

$$(x - a)^{n+1} \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

$f$

$(\xi)$ , with  $\xi \in (a, x)$ .

$n$

$(n + 1)!$

$1$

We now wish to extend this theorem to functions of two variables. Specifically,

we wish to expand a function  $z = f(x, y)$  of two variables in a Taylor series

about a point  $(a, b)$

$(a, b)$  in the domain

$D$

$D \subset \mathbb{R}^2$  of the function.

Let  $z = f(x, y)$  be a function of two variables which is continuous, together with

all its partial derivatives up to the

$n$ th

$(n + 1)$

order inclusive, in some neighborhood

of the point  $(a, b)$ .

Then, like in the case of a function of a single variable, the function  $f(x, y)$  can be represented in the form of a sum of an  $n$ -th degree polynomial in powers of  $(x - a)$ ,  $(y - b)$ , and a remainder term  $R$ .

$n$

Under this assumption the function  $f(x, y)$  can be expressed in a Taylor formula about the point  $(a, b)$

as follows:

$k$

$n + 1$

$\partial$

$\partial$

$f(x, y) =$

$(x - a)$

$+ (y - b)$

$\sum$

)

$f(a, b)$

$+ R,$

$n$

$k + 1$

!

$= k$

$x$

$\partial$

$y$

$\lfloor$

$\partial \rfloor$

$n$

$1$

$(+)$

$1$

$\lceil$

$\partial$

$\partial \rceil$

where  $R =$

$(x - a)$

$+ (y - )$

$b$

$f(\xi, \eta)$

$n$

$(n + )!$

$1 \mid$

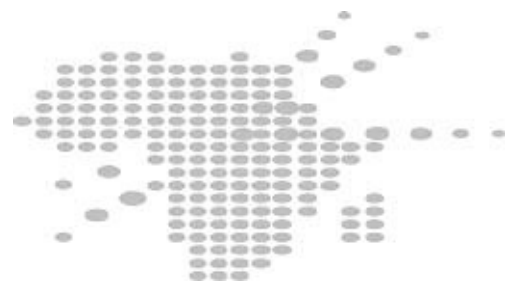
$x$

$\partial$

$y \mid$

$\lfloor$

$\partial \rfloor$



## African Virtual University 112

for some point  $(\xi, \eta)$  lying in the disc centered at  $(a, b)$

and containing the

point  $(x, y)$ .

### Observation

The learner should pay special attention to the operator

$$D = (x - a) \frac{\partial}{\partial x} + (y - b) \frac{\partial}{\partial y}$$

– )

$$+ (y - b) \frac{\partial}{\partial y}$$

– )

which appears in the above Taylor expansion

$x$

$\frac{\partial}{\partial x}$

$y$

$\frac{\partial}{\partial y}$

formula. Powers of this operator are formed in accordance with the binomial

expansion of the term

$n$

$(a + b)^n$ . Typically,

2

2

2

2

2

[

$\partial$

$\partial ]$

2  $\partial$

$\partial$

2

**D = (x - a)**

**+ (y - )**

**b**

**= (x - a)**

**+ 2(x - a)(y - )**

**b**

**+ (y**

**)**

**b  $\partial$**

**-**

**|**

**|**

2

2

**x**

$\partial$

**y**

$\partial$

$x$

$\lfloor$

$\rfloor$

$\partial$

$x$

$\partial y$

$\partial$

$y$

$\partial$

**Do This**

**Write down all the terms of the differential operator  $D^3$ .**

**Maximum, Minimum and Saddle Points Functions of Two Variables**

**Let  $z = f(x, y)$  be a function of two variables defined over a region**

**2**

**$D \subset \mathcal{R}$**

**We begin by making three observations:**

**1. The maximum and minimum values of a function in a given domain are collectively referred to as extreme values or simply as extrema of the function.**

**Without specifying the nature of the extreme value, the singular form of the word extrema is extremum.**

**2. An extremum of a function  $z = f(x, y)$  may be absolute or relative.**

**3. An extremum of a function  $z = f(x, y)$  can occur only at**

**i) a boundary point of the domain of  $f$ ,**

**ii) at interior points where  $f_x = f_y = 0$  or**

**x**

**y**

**iii) at points where  $f$  and  $f'$  fail to exist .**

**x**

**y**



**African Virtual University 113**

**Definitions**

**i) A point  $P(x,$**

**is said to be a point of absolute maximum if**

**$y)$**

**$P \in D$**

**$f(P) \geq f(Q)$  for all points  $Q \in D$  .**

**ii) A point  $P(x,$**

**is said to be a point of absolute minimum if**

**$y)$**

**$P \in D$**

**$f(P) \leq f(Q)$  for all points  $Q \in D$  .**

**iii) A point  $P(x,$**

**is said to be a point of absolute maximum if**

**$y)$**

**$P \in D$**

**$f(P) \geq f(Q)$  for all points  $Q \in D$  which are in a small neighborhood**



**of point**

**iv) A point  $P(x,$**

**is said to be a point of absolute minimum if**

**$0 < y <$**

**$0 < x <$**

**$f(P) \leq f(Q)$  for all points  $Q \in D$  which are in a small neighborhood**

**of point**

**As can be noted from the above definitions, the difference between absolute and relative maximum or minimum lies in the set of points  $Q$  one compares the fixed point  $P$  with. If  $Q$  is restricted to a small neighborhood of point  $P$  then we speak of relative extrema (minimum or maximum).**

**Basic Question:**

**How can one locate the points  $P$  at which the function**

**$f(x, y)$  has extreme values?**

**Assuming that  $f(x, y)$  has continuous first partial derivatives, the answer to the above question is simple:**

**All maximum and minimum points which lie in the interior of the domain of definition of  $f(x, y)$  must be among the critical points of the function.**

**The critical points of  $f(x, y)$  are those points which simultaneously satisfy the two equations**

**$\frac{\partial f}{\partial x}$**

**$= 0$  ;**

**$\frac{\partial f}{\partial y}$**

**$= 0$  ;**

**$\frac{\partial f}{\partial z}$**

$\partial x$

$y$

$\partial$

Once we have obtained all the critical points we shall then have to classify which among them are maximum points and among them are minimum points. We shall also have to say something about any critical points which represent neither a maximum nor a minimum point of the function. An easy method of classifying



African Virtual University 114

critical points involves the second partial derivatives of  $f(x, y)$ . The method is given in Section 2.4 above and for ease of reference, we have reproduce it here.

### Second Derivative Test for Local Extreme Values

Let  $f(x, y)$  possess continuous first and second partial derivatives at a critical point

$P(x, y)$ . At  $P$  we define the quantity

$0$

$0$

$f$

$f$

$xx$

$xy$

$2$

$D(a, )$

$b =$

$= f f - f$

.

$xx yy$

$xy$

$f$

$f$

$yx$

$yy$

• If  $D > 0$  and  $f$

, then  $f(a, )$

$b$  is a local minimum.

$xx > 0$

• If  $D > 0$  and  $f$

, then  $f(a, )$

$b$  is a local maximum.

$xx < 0$

• If  $D < 0$  then  $f$  has neither a local minimum nor a local maximum point

at  $(a, )$

$b$ . The point is known as a saddle point.

**Worked Example**

Classify the critical points of the function  $f(x, y) = x^3 + y^3 - 3xy$ .

**Solution**

The first and second partial derivatives are

$\partial f$

$f$

$\partial$

$$= 3x^2 - 3y;$$

$$= 3y^2 - 3x$$

$\partial x$

$y$

$\partial$

$2$

$\partial f$

$\partial^2 f$

$\partial^2 f$

$2$

$\partial f$

$$= 6x;$$

$=$

$$= -3;$$

$$= 6y.$$

$2$

$x$

$\partial$

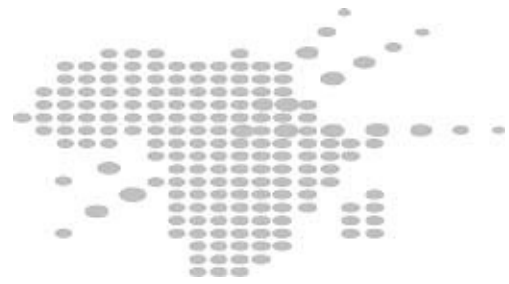
$\partial x \partial y$

$\partial y \partial x$

$2$

$y$

$\partial$



**African Virtual University 115**

$\partial f$

**f**

$\partial$

**Solving the simultaneous nonlinear equations**

**= 0 ;**

**= 0 one gets**

$\partial x$

**y**

$\partial$

**the following critical points: (0,0) and**

**)**

**1**

**,**

**1**

**(**

**. Applying the second derivative**

**test on each point we find:**

**0**

**- 3**

**For point P (0,0) : D( ,**

$$f''(0,0) =$$

$$= 9$$

$$- < 0$$

$$- 3$$

$$0$$

**We therefore conclude that the point P (0,0) is neither a local minimum nor a local maximum, but is a saddle point.**

$$6$$

$$- 3$$

**For point P**

$$)$$

$$1$$

$$,$$

$$1$$

$$($$

$$: D$$

$$)$$

$$1$$

$$,$$

$$1$$

$$($$

$$=$$

$$= 25 > 0$$

$$- 3$$

$$6$$

Since  $f$

)

1

,

1

(

we conclude that  $P$

)

1

,

1

(

is a point of relative minimum.

$xx$

$> 0$

**Self Exercise**

**1. Test the surface**

4

4

$z = f(x, y)$

$= 4xy - x - y$  for maxima, minima and saddle

points. Calculate the values of the function at the critical points.

**2. Determine the relative extrema for the function**

$f(x, y)$

$(x, y)$

3

3

$$= x + y + 3z$$

$$y - 3x - 9y + 2$$



African Virtual University 116

## Lagrange Multipliers

### Constrained Extrema For Functions Of Two Variables

In Section 2.5.3 we discussed the problem of determining the extrema (maximum and minimum values) of a function  $f(x, y)$ . We did so without imposing any condition on the set  $D$  of possible values of  $x$  and  $y$ .

In this Section we revisit the problem of finding an extremum (singular form of extrema) of a function  $f(x, y)$  but with a difference. Here we impose a condition (or a set of conditions) on the variables  $x$  and  $y$ . Typically we may wish to find the extrema of  $f(x, y)$  only for points lying on a particular curve in the domain  $D$  of the function, such as

$$g(x, y) = 0$$

The problem of finding extreme values of a function  $f(x, y)$  subject to a constraint of the general form  $g(x, y) = 0$  is a typical example of a constrained optimization problem.

#### Example

Find the maximum volume of a rectangular box having a square base and open at the top, if the side length of the base is  $x$  cm, the height is  $y$  cm and the total



area of the material to be used is 108

2

cm

### Reduction Method

A possible method of solving the general constrained minimization problem posed here is the following:

(i) Solve the equation  $g(x, y) = 0$  for one of the variables, whichever is most convenient to solve for.

(ii) Substitute the result into the function  $f(x, y)$  being optimized. The substitution will eliminate one of the variables from the expression for  $f(x, y)$  reducing the problem to a single variable extremum problem.

To illustrate this we apply the method on the example given above.



## Volume to be maximized:

$$V(x, y) = x^2 y$$

Area of the box:

$$A(x, y) = x^2 + 4xy$$

Constraint:

$g(x, y)$

$x^2 + 4xy$

$-108 = 0$

$$x^2 + 4xy - 108 = 0$$

$108 - x^2$

$-$

We solve  $g(x, y) = 0$  for  $y$  and get

$y =$

. We substitute this

$4x$

expression for  $y$  into  $V(x, y)$  to eliminate  $y$  and get

$$V(x) = x^2 \left[ \frac{108 - x^2}{4x} \right]$$

$1$

$2$

$3$

$$V(x) = x^2 \left[ \frac{108 - x^2}{4x} \right]$$

$=$

$3$

$$27x - x^3$$

[

4 x

]

4

The critical points of  $V(x)$  are found to be  $x = \pm 6$ . Since  $x$  represents a length, we take the positive value  $x = 6$ . The corresponding value of  $y$  is  $y = 3$ . The maximum volume of the box subject to the given constraint is therefore

$$V = 108$$

3

cm .

### Method of Lagrange Multipliers

The reduction method we have illustrated here is applicable only to those problems where it is possible to solve the equation  $g(x, y) = 0$  for one of the variables.

We now present another method which is applicable more generally.

The method of Lagrange multipliers involves the introduction of an additional independent variable  $\lambda$  called the Lagrange Multiplier.

The essential stages in its application are the following:

1. Using the functions  $f(x, y)$ ,  $g(x, y)$  and  $\lambda$  we define a new function

$$F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

2. We then find the critical points of  $F(x, y, \lambda)$  by solving the three simultaneous equations

$$\frac{\partial F}{\partial x}$$

$$\frac{\partial F}{\partial y}$$

$$\frac{\partial F}{\partial \lambda}$$

$$(x, y, \lambda) = 0;$$

( ,

x ,

$$y \lambda) = 0 ;$$

$$(x, y, \lambda ) = 0.$$

$\partial x$

$\partial y$

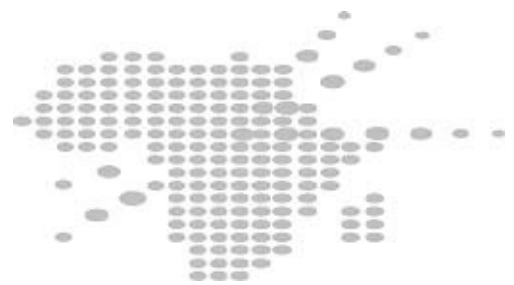
$\partial \lambda$

**3. Assuming that we have obtained all the critical points (x , y ,**

**we then**

**k**

**k  $\lambda$  k )**



**African Virtual University 118**

**list the points (x , y and note that they automatically satisfy the constraint**

**k**

**k )**

$$g(x, y) = 0 .$$

**4. We finally compare the values f (x , y ) of f (x, y) at the points (x , y**

**k**

**k )**

**k**

**k**

**The largest (smallest) value is the absolute maximum (minimum) on the curve**

$$g(x, y) = 0.$$

### Example

We shall demonstrate the Lagrange Multipliers method on the example above which we had solved using the method of elimination of one of the variables in

$f(x, y)$  using the constraint equation  $g(x, y) = 0$ .

With  $f(x, y) = V(x, y) = x^2 y$  and  $g(x, y) = 2$

$x + 4xy - 108 = 0$ , the

expression for the function  $F(x, y, \lambda)$  is:

$$F(x, y, \lambda) = x^2 y - \lambda(2$$

$x + 4xy -$

)

108.

The critical points of  $F(x, y, \lambda)$  are obtained by solving the three simultaneous equations

$\partial F$

$$= 2xy - 2x\lambda - 4y\lambda = 0$$

$\partial x$

$\partial F$

$$= x^2 - 4x\lambda = 0$$

$\partial y$

$\partial F$

=

2

$$-x - 4xy + 108 = 0$$

$\partial \lambda$

3

The solution of this system is  $x = 6$ ,  $y = 3$  and  $\lambda =$

, a result which yields

2

the same maximum value 108 for the volume  $V$  of the rectangular tin.

### Self Exercise

1. Maximize the function  $f(x, y) = xy$  along the line  $x + 2y - 2 = 0$  by applying both the method of eliminating one of the variables in  $f$  as well as applying the Lagrange multipliers method.



African Virtual University 119

2. Apply the Lagrange Multipliers method to minimize the function 2

2

$x + y$

subject to the constraint  $g($

$x y) = 3x + 4y - 25 = 0$  .

Exercise Questions (From Cain and Herod: Multivariable Calculus)

### Exercise

Solve all the exercise problems given in the unit. (Chapters 7 and 8)



XVI. Summative Evaluation

Summative Assessment Questions

Unit 1: Elementary Differential Calculus

1. Given the function

$$f(x)$$

$$2$$

$$= x - 2x + 3, \text{ find:}$$

$$f(2 + h) - f(2)$$

$$2$$

$$f(x + h) - f(x)$$

(a)

$$\left[ \frac{f(2 + h) - f(2)}{h} \right]$$

$$-$$

$$\left] \lim_{h \rightarrow 0} \right.$$

lim

$$\left[ \frac{f(x + h) - f(x)}{h} \right]$$

$$-$$

$$\left] \lim_{h \rightarrow 0} \right.$$

$$\left| \frac{f(x + h) - f(x)}{h} \right|$$

$$\left| \lim_{h \rightarrow 0} \right|$$

(b)

$$\lim_{h \rightarrow 0} \left| \frac{f(x + h) - f(x)}{h} \right|$$

$$\left| \lim_{h \rightarrow 0} \right|$$

$$\rightarrow$$

$h \rightarrow 0$

$h$

]

$\rightarrow$

$h \rightarrow 0$

$h$

]

2. Without applying L'Hopital's Rule, find the following limits if they exist.

[ 3

$x - 1$  ]

[ 3

$x - 1$  ]

(a)  $\lim$  |

|.

(b)

$\lim$  |

|.

$x \rightarrow 1$

$x \rightarrow 1$

$x \rightarrow 1$

$\rightarrow$

[

- ]

-

$x \rightarrow 1$



$|| - ||$

1

(c)

[

]

lim |

.

2 |

x

x

→ 0

[

]

3. Evaluate the following limits

[

2

x + 1 ]

[

3

x - 2

]

(a) lim|

(b)

.

3

|

lim | 3

|

$x \rightarrow \infty [3x + 10 ]$

x

3x

2 x 3

$\rightarrow \infty [$

+

- ]

sin x

cos x

(c)

[

]

lim

[

]

|

|.

(d)

lim|

|

x

x

**x**

**x**

→ ∞ [

]

→ ∞ [

]



**African Virtual University 121**

[ a

|

**for**

**x ≤ 0**

| x - 2

**4. Given that**

**f(x) = { 2x + b for 0 < x < 2**

|

**6**

**for**

**x ≥ 2**

|

]

**for what values of a and b is f continuous over ℝ**

[ tan( x ) ]

## 5. Apply L'Hopital's rule in calculating

lim

·

$\pi$  |

|

x

$\tan(3x)$

→

[

]

2

## Unit 2: Elementary Integral Calculus

6. Using the method of substitution, find the anti-derivative of each of the following functions:

1

(a)  $f(x) = \ln(x)$

x

x

e

(b)  $f(x) =$

x

$1 - e$

$\sin(x)$

(c)  $f(x) = \cos(x)$

7. Apply the method of integration by parts in finding anti-derivatives and hence

evaluate the integrals

1

(a)  $\int_0^1 x e^x dx$

0

2

(b)  $\int_1^{\pi} \ln(x) dx$

1

$\pi$

(c)  $\int_0^{\pi/2} x \cos(x) dx$

0



African Virtual University 122

$\pi$

8. Let  $I_n$

$=$

$=$

$\int_0^{\pi/2} \sin^n(x) dx$ .

$n$

0

(a) Find

$I_n$ .

0

(b) By expressing the integrand in the form  $\sin^n(x) = \sin(x) \sin^{n-1}(x)$

-

=

(x)

n - 1

and integrating by parts, show that I

I

n =

n - 2

n

$\pi$

(c) Use the above results to evaluate .

$\int_0^{\infty} \sin^2(x) dx$

$\int_0^{\infty}$

$\infty$

1

dx

9 (a)

Explain briefly why the integrals

$\int_0^{\infty} e^{-x} dx$

$\int_0^{\infty} e^{-x^2} dx$

,  $\int_0^{\infty} e^{-x^2} dx$

are called

0

0

1

**improper integrals.**

**- x**

**(b)**

**In each case, determine whether or not the improper integral converges.**

**Where there is convergence, find the value of the integral.**

**10. Approximate the following integrals using both the trapezium rule and the**

**Simpson rule. In each case, evaluate the integrand correct to four decimal**

**places and use two interval lengths:  $h = 0.25$  and  $h = .$**

**0 1**

**1**

**dx**

**2**

**2**

**dx**

**(a)**

**∫**

**(b) ∫ ln(x)dx (c) ∫**

**0**

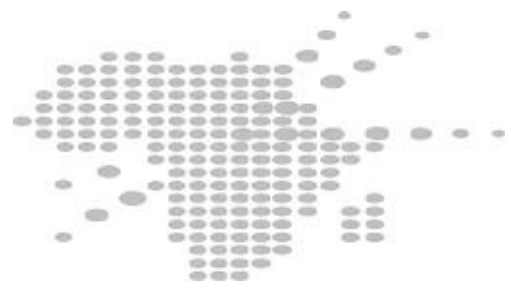
**1 + x**

**1**

**1**

**+ 2**

**1 x**



**African Virtual University 123**

**Unit 3: Sequences and Series**

**11. (a) Define a Cauchy sequence.**

$\infty$

**1**

**(b) Prove that the sequence of partial sums of the harmonic series  $\sum$  is**

**k**

**k**

**=1**

**not a Cauchy sequence. What conclusion follows from this result with regard to convergence of the series?**

$\infty$

**12. (a) Write down the th**

**n partial sum of the series  $\sum a$  .**

**k**

**k =1**

**(b) How is the convergence of a series related to its n – th partial sum?**

$\infty$

**1**

**(c) Find the n – th partial sum of the series  $\sum$**

**and hence**

**k**



$k$   
 $k$   
(  
)(  
1  
2)  
1  
+  
+  
=

find the sum of the series, if it converges.

$\infty$

13. (a) What is meant by saying that the series  $\sum_{k=1}^{\infty} a_k$  converges

$k$   
 $k = 1$

(i) Absolutely,

(ii) Conditionally.

(b) Prove that if a series converges absolutely, then it converges.

14. By applying the integral test determine the values of the parameter  $p$  for

$\infty$   
1

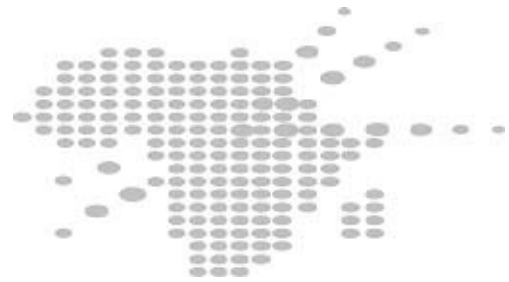
which the series  $\sum_{k=1}^{\infty} \frac{1}{k^p}$  converges.

$p$   
 $k$

$k = 1$

15. Expand  $\cos(x)$  in a Taylor series about the origin, giving the first three nonzero terms.

Using the sigma notation write down the series for the function. Determine both the interval of convergence and the radius of convergence of the resulting series.



African Virtual University 124

Unit 4: Calculus of Functions of Several Variables

16. (a) Give the  $(\epsilon, \delta)$  - definitions of the limit and continuity concepts for a function  $f(x, y)$  at an interior point  $(a, b)$  in the domain  $D$  of  $f$ .

[

$2xy$

]

(b) Find

$\lim$

|

if it exists or show that the limit does not

2

|

$(x, y) \rightarrow$

2

$(0, 0)$

exist.

$$[x + 2y]$$

2

2

$\partial T$

$\partial T$

17. A function  $T(x, y)$  satisfies the equation

+

$= 0$ . Find the

2

2

$\partial x$

$\partial y$

corresponding equation in polar coordinates  $x = r \cos \theta$ ,  $y = r \sin \theta$ .

18. Find the centre of mass (center of gravity) of a lamina with density function

2

$f(x,$

$y) = 1 + 2y + 6x$  bounded by the region  $R$  enclosed between the two

curves

2

$y = x$ ,

$y = 4$

19. Find all the critical points of the function

2

2

$f(x, y)$

$$-x - y$$

$$= xye$$

and apply the

second derivative test to classify them.

$\partial$

$\partial$

20. (a) If  $D$  is the differential operator

$h$

$+ k$

, find

$\partial x$

$\partial y$

2

$D f(x, y)$ ,

and

3

$D f($

$x y)$ .

(b) Give the first three terms of the Taylor series expansion of the function

$xy$

$f(x, y) = e$  about the point  $(2,3)$ .



## Summative Assessment Answers

1. (a)

2.

(b)

$$2x - 2$$

2. (a)

3

(b)

0

(c)

Limit does not exist

3. (a)

Limit does not exist ( $\infty$ )

1

(b)

3

(c)

0

(d)

0

4.  $a = 4$  and

$$b = 2.$$

5. 3.

1

6. (a)  $\ln 2(x) + C$

2

(b)

x

$$C - 2 \ln e$$

(c)  $C - \ln[\cos(x)]$

7. (a)

-1

(b)

$$2 \ln(2) - 1$$

$\pi$

(c)

-1

2



African Virtual University 126

8. (a)

$$I_0 = \pi$$

$\pi$

(c)

$$\sin 2$$

$\pi$

(x)

=

$$\int_0^{\infty} dx$$

9. (a)

$$\int_0^{\infty} e^{-x} dx$$

$\int_0^{\infty} e^{-x} dx$  is an improper integral because the upper limit of integration

0 is not finite.

$$\int_0^1 dx$$

$\int_0^1 \frac{1}{x} dx$  is an improper integral because the integrand is not defined at

$$\int_0^1 \frac{1}{x} dx$$

$$\int_0^{\infty} e^{-x} dx$$

(b)

$$\int_0^{\infty} e^{-x} dx$$

$$\int_0^{\infty} e^{-x} dx$$

**is divergent.**

**0**

**1**

**dx**

**∫**

**converges to 2 .**

**0**

**1 - x**

**10.**

**Integral**

**Trapezium Rule**

**Simpson's Rule**

**h = 0.25 h = 0.1**

**h = 0.25 h = 0.1**

**1**

**dx**

**∫**

**0.8301**

**0.8287**

**0.8284**

**0.8285**

**1 x**

**0**

**+**

**2**



$$\int \ln(x) dx$$

$$0.3837$$

$$0.3857$$

$$0.3862$$

$$0.3863$$

$$1$$

$$2$$

$$dx$$

$$\int$$

$$0.3235$$

$$0.3220$$

$$0.3218$$

$$0.3217$$

$$2$$

$$1 x$$

$$1$$

$$+$$



**African Virtual University 127**

**11. (a) A sequence  $\{ a_n \}$  is called a Cauchy Sequence if for every positive number  $\epsilon > 0$  however small, a corresponding positive integer  $N(\epsilon)$  exists, such that a**

**a**

**for all ,**

**$m n > N (\epsilon ) .$**

**$m -$**

**$n < \epsilon$**

**1**

**(b) With a =**

**, we let  $m = 2 n$  and note that however small one chooses**

**k**

**k**

**1**

**1**

**1**

**1**

**[ 1 ]**

**1**

**an  $\epsilon > 0 , S$**

**for all**

**$m - S n =$**

**+**

**+**

**+ ... +**

**> n**

**=**

**$n n + 1 n + 2$**

2

| |

n

[n]

2

$n > N(= 2n)$ .

$\infty$

12. (a) The  $n$ -th partial sum of the series  $\sum a$  is

k

k=1

n

S =

a = a

.

1 + a<sup>2</sup> + a + ...

3

+ a

n

$\sum k$

n

k=1

$\infty$

(b) The series  $\sum a$  converges if, and only if,  $\lim S$  exists.

k

n

$n \rightarrow \infty$

$k = 1$

1

1

1

(c) Since

=

-

, it follows that

$(k + )$

$1 k + )$

2

$k + 1 k + 2$

$n / 1$

$1 \setminus [ ( 1 1 \setminus ) ( 1 1 \setminus )$

$( 1$

$1 \setminus ] / 1$

$1 \setminus$

S

$n = \sum |$

-

$| = | - | + | - | + \dots + |$

-

$| = | -$

|

|

|

k

k 1 k 2

2

3

3

4

n 1 n 2

2

n 2

= \

+

+ ) \ \

) \

)

\ +

+ ) \ \

+ )

1

∞

1

and hence  $\lim S$

. Therefore, the infinite series

conver-

$n =$

$\Sigma$

$n \rightarrow \infty$

2

k

k

k

(

)(

1

2)

1

+

+

=

1

ges to .

2

$\infty$

13. (a) (i) An infinite series  $\Sigma a_k$  converges absolutely if the series

k

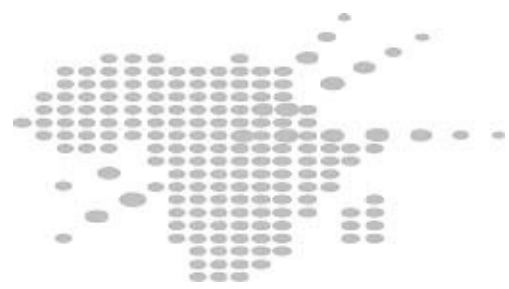
k=1

$\infty$

$\Sigma a_k$  of positive terms converges.

k

$k = 1$



**African Virtual University 128**

$\infty$

**(ii)**

**An infinite series  $\sum a_k$  that converges but is not**

**$k$**

**$k$**

**absolutely**

**$= 1$**

**Convergent is said to be conditionally convergent.**

**(b) Sketch of proof that an absolutely convergent series is convergent:**

**Assume that the series converges. Note that for all  $k$ ,**

**$0 \leq a_k + a_k \leq 2 a_k$ , and by the comparative test it follows that the**

**$k$**

**$k$**

**$k$**

$\infty$

$\infty$

**series  $\sum (a_k$**

**$a_k$**

**converges. Since**

**$\sum a_k$  also converges, then the**

$k +$

$k )$

$\sum k$

$k = 1$

$k = 1$

$\infty$

$k$

series  $\sum ($

$\{a a$

$a$

$a$  is also convergent.

$k -$

$k )-$

$k \} = \sum k$

$k = 1$

$k = 1$

$\infty$

$1$

**14. Application of the integral test on the series  $\sum$**

**. One analyses the**

**p**

**x**

**k = 1**

**$\infty dx$**

**convergence properties of the improper integral  $\int$ .**



**p**

**x**

**1**

**b**

$\infty dx$

**b**

**1**

**dx**

[ -

x p ]

1 [

1

]

∫

= lim

= lim

=

lim

**b**

∫ p

|

|

(

-

**p ≠**

$\rightarrow \infty$

$\rightarrow$

$\mathbf{b} \infty$

|

$\mathbf{p} \mathbf{1}$

)

$\mathbf{1},$

.

$\mathbf{1}$

$\mathbf{p}$

$\mathbf{x}$

$\mathbf{x}$

$\mathbf{1-p}$

$\mathbf{1}$

-

$\rightarrow$

$\mathbf{b} \infty$

|

$-\mathbf{p}$

$\mathbf{b}$

$\mathbf{1}$

$\mathbf{1}$

|

$\mathbf{1}$

|

]

$\infty$

1

The limit exists only if  $p > 1$ . Therefore, the  $p$ -series  $\sum$   
converges

$p$

$x$

$k = 1$

for  $p > .$

1

1

1

15. (a)

2

4

$\cos(x) = 1 - x +$

$x .$

2

24

$\infty$

2 k

$x$

(b)  $\cos(x) = \sum (-)$

1 k

. The interval of convergence is  $-\infty < x < \infty$

**k**

**k**

**2!**

**= 0**

**the radius of convergence is  $R = \infty$ .**



**African Virtual University 129**

**16(a) Definition of limit of  $f(x, y)$  at point  $(a, b)$   $f(x, y)$  is said to have limit  $L$  as  $(x, y)$  tends to (or approaches)  $(a, b)$  if for every  $\epsilon > 0$  there exists a  $\delta = \delta(\epsilon)$  such that  $f(x, y) - L < \epsilon$  whenever**

**$\sqrt{(x - a)^2$**

**0**

**$+ (y - b)^2$**

**$< \delta$ . One writes**

**lim**

**$f(x, y) = L$ .**

**$(x, y) \rightarrow (a, b)$**

**Definition of continuity of  $f(x, y)$  at point  $(a, b)$**

**$f(x, y)$  is continuous at  $(a, b)$  if for every  $\epsilon > 0$  there exists a  $\delta = \delta(\epsilon)$  such that  $f(x, y) - f(a, b) < \epsilon$  whenever**

**$\sqrt{(x - a)^2 + (y - b)^2} < \delta$**

**$< \epsilon$**

**2**

**2**

**0**

**x a**

**y b**

**whenever  $\sqrt{(x-a)^2 + (y-b)^2} < \delta$ .**

**16(b) We try a linear approach to the origin by setting  $y = mx$  and substituting**

**$\sqrt{x^2 + y^2}$**

**$\sqrt{x^2 + m^2 x^2}$**

**into the function  $f(x, y) =$**

**to give the expression**

**for**

**$\sqrt{x^2 + m^2 x^2}$**

**$\sqrt{x^2 + m^2 x^2}$**

**$x + 2y$**

**$\sqrt{x^2 + m^2 x^2}$**

**$1 + 2m$**

**the function away from the origin. Since the value of this expression depends on**

**the choice of the slope  $m$  we conclude that the limit is not unique, and therefore**

**the function has no limit as one approaches the origin.**

**$\sqrt{x^2 + m^2 x^2}$**

**$\sqrt{x^2 + m^2 x^2}$**

**$\partial T$**

**$1 \partial T$**

**$1 \partial T$**

**17.**

**+**

**+**

2

2

2

$\partial r$

$r \partial r$

$r$

$\theta$

$\partial$

1047

18. The coordinates of the center of mass are:  $X = 0, Y = 378$

19. The critical points are the solutions of the two simultaneous nonlinear

$\partial f$

$\partial f$

equations

$= 0,$

$= 0$

$\partial x$

$\partial y$

One finds the following 5 critical points:

(

$\sqrt{1}$

$1 \sqrt{1}$

1

$1 \sqrt{1}$

$1 \sqrt{1}$

1

1 \

0,0), |

,

|, | -

, -

|, |

, -

|, | -

,

|

\ 2

2 / \

2

2 / \ 2

2 / \

2

2 /



**African Virtual University 130**

**Classification:**

( 1

1 \

**Saddle point:**

**(0,0)**

**Local maxima:  $\pm$  |**

,

|

\ 2

2 )

(

1

1 \

**Local minima:  $\pm$  | -**

,

|

\

2

2 )

2

2

2

$\partial \mathbf{f}$

$\partial \mathbf{f}$

$\partial \mathbf{f}$

**20. (a)**

2

2



2

$D f (,$

$x y) = h$

$+ 2hk$

$+ k$

,

2

2

$x$

$\partial$

$x$

$\partial y$

$\partial$

$y$

$\partial$

3

3

3

3

$\partial f$

$\partial f$

$\partial f$

$\partial f$

3

3

2

2

3

$D f (,$

$x y) = h$

$+ 3h k$

$+ 3hk$

$+ k$

3

2

2

3

$x$

$\partial$

$x$

$\partial$

$y$

$\partial$

$x$

$\partial y$

$\partial$

$y$

$\partial$

(b)

6 [

$$\begin{aligned}
 & 9 \\
 & 1 \\
 & \text{exy} = e^{1+} ( \\
 & 3x - ) \\
 & 2 + ( \\
 & 2y - ) \\
 & 3 + (x - 2)^2 + (x - 2)(y - ) \\
 & 3 + ( \\
 & 2y - 3)^2 ] \\
 & | \\
 & | \\
 & \cdot \\
 & [ \\
 & 2 \\
 & 7 \\
 & ]
 \end{aligned}$$



**African Virtual University 131**

## **XVII. Main Author of the Module**

### **Biography Of The Author**

**Dr. Ralph W.P. Masenge is Professor of Mathematics in the Faculty of Science, Technology and Environmental Studies at the Open University of Tanzania. A graduate of Mathematics, Physics and Astronomy from the Bavarian University**

**of Wuerzburg in the Federal Republic of Germany in 1964, Professor Masenge obtained a Masters degree in Mathematics from Oxford University in the United Kingdom in 1972 and a Ph. D in Mathematics in 1986 from the University of Dar Es Salaam in Tanzania through a sandwich research program carried out at the Catholic University of Nijmegen in the Netherlands.**

**For almost 33 years (May 1968 – October 2000) Professor Masenge was an academic member of Staff in the Department of Mathematics at the University of Dar Es Salaam during which time he climbed up the academic ladder from Tutorial Fellow in 1968 to full Professor in 1990. From 1976 to 1982, Professor Masenge headed the Department of Mathematics at the University of Dar Es Salaam.**

**Born in 1940 at Maharo Village, situated at the foot of Africa's tallest mountain, The Killimanjaro, Professor Masenge retired in 2000 and left the services of the University of Dar Es Salaam to join the Open University of Tanzania where he now heads the Directorate of Research and Postgraduate Studies, chairs the Senate Committees of Research, Publications and Consultancy Committee and the Postgraduate Studies Committee, besides coordinating a number of mathematics courses in Calculus, Mathematical Logic and Numerical Analysis.**

**Professor Masenge is married and has four children. His main hobby is working on his small banana and coconut garden at Mlalakuwa Village situated on the outskirts of Dar Es Salaam, the commercial capital of the Republic of Tanzania.**

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